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Chapter E

E.I Introduction

The algebraic part of Ehrenpreis' Fundamental Principle states roughly that a system of linear partial differential equations, with complex constant coefficients, is fully determined by its exponential solutions.

A reciprocal point of view turns the action around: instead of having the polynomial (differential operator) act on the exponentials we may view the exponentials as parametrized differential operators acting back on the polynomial. Palamodov calls these operators *Noetherian Operators*, presumably in honor of Max Noether and the famous Noether Conditions in curve theory.

If the symbols of the operators involved generate, e.g., a primary ideal $\mathbf{q}$, then the Ehrenpreis Principle may be stated as a membership criterion. A $\mathbf{p}$–primary ideal may be characterized by the Noetherian operators annihilating it on the variety belonging to it, i.e., on reduction modulo $\mathbf{p}$. Ehrenpreis speaks of "multiplicity varieties", varieties equipped with a Noetherian operator. An algebraic geometer would speak of "affine schemes".

These theorems were proved by Ehrenpreis and Palamodov, in [EHR] and [PAL]. Unfortunately, their proofs are hard to follow, or even locate. There are cleaner proofs in [HÖR] and [BJÖ]. Hörmander's treatment is arithmetic in flavor, utilizing normalizations and discriminants. Björk's proof uses analytic completion and flatness arguments, thereby inverting all discriminants and Jacobians.

In this note we offer a purely algebraic proof, requiring, of course, an algebraic definition of (generic) exponential solution. The duality statement above is then deduced as a special case of Matlis' duality theory, which may be found in [MAT], or the textbook [MAS].

The main step is to describe the injective envelope $E(A/\mathbf{p})$ of the ring $A/\mathbf{p}$, where A is the polynomial ring $k[x_1, x_2, \ldots, x_n]$, (k a field of characteristic 0) and $\mathbf{p}$ is a prime ideal.

I give two descriptions. The first depends on a Noetherian normalization and is the one required here. The second does not depend on normalization. In it injective envelopes are described as the highest cohomology of a de Rham complex; this actually gives a means of comparing different normalizations. These are my main results, and I believe at least the second one is new.

I have provided proofs of some well-known or at least folk-lore results, whenever a suitable reference was lacking, or whenever special assumptions allowed more transparent proofs than the ones usually given.

Towards the end of the paper I connect Noetherian Operators with the better known concept of residues. They are *pre-residues*, the "pre" signifying an intermediate result before taking traces from the quotient field K of $A/\mathbf{p}$ to a normalization $k(\tau')$.

In order not to delay this presentation further I offer this discussion in a rather sketchy and incomplete stage. I hope to return to these matters in a later paper, along with applications to Elimination Theory. I have also saved a discussion of fundamental classes (with a construction that I believe is new in the non-perfect case) for a later communication. Therefore I will not discuss regularity questions in this paper, i.e., all operators and residues are allowed to have their coefficients in the full quotient rings.

The plan of the paper is

I) Introduction
II) Wiebe Duality
III) Regular Prime Bases
IV) Generic and True Exponentials
V) de Rham Cohomology and Trace
VI) E($A/\mathbf{p}$), First Determination
VII) E($A/\mathbf{p}$), Second Determination
VIII) Noetherian Operators, Ehrenpreis Duality
IX) Residues (sketch)
X) Examples

Notation

"Ring" will always mean "commutative Noetherian ring with 1". "Mapping" will almost always mean "R-linear mapping", R being a ring.

k will denote an algebraically closed field of characteristic 0.

$\mathbf{p}, \mathbf{m}$ will be prime ideals, $\mathbf{m}$ mostly maximal

$\mathbf{q}$ is a primary ideal (usually belonging to $\mathbf{p}$).

Sometimes quite general ideals will be denoted I, J.

Sometimes these letters will denote index sets. No confusion is possible. The statements $I \leq J$, $J \pm I$ are to be interpreted as component-wise inequality, addition, etc. $I < J$ means "$I \leq J$ and $I \neq J$". f^I refers to the sequence $f_k^{i_k}$, or the product of these element; $(f)^I$ is the ideal generated by that sequence. df is $\ldots \wedge df_k \wedge$, df/f is $df/\Pi_k f_k$.

A is the polynomial ring $k[X_1, X_2, \ldots X_n] = k[X]$.

$R = A/\mathbf{p}$ as ring or A-module.

K is its quotient field, $(A/\mathbf{p})_\mathbf{p}$

Q denotes full quotient ring, e.g., $Q(A/\mathbf{q}) = (A/\mathbf{q})_\mathbf{p}$.

For the residue classes of the X_i, in R, we write τ_i. A $'$ will distinguish parameter variables with respect to a Noetherian normalization, a $''$ the fiber variables:

$$R = k[\tau] = k[\tau'_1, \ldots, \tau'_d][\tau''_{d+1}, \ldots \tau''_n]$$

Often we will have to use $x'_1, \ldots, x'_d$ instead of the τ'_i. However, x''_j will always denote indeterminates or their residue classes modulo some $\mathbf{p}$-primary ideal $\mathbf{q}$. Sometimes other symbols will be used.

I will often have to consider objects like $g(\tau', \tau'')$ and $g(\tau', x'')$ side by side, or even $h(\tau', \tau'', x'')$. In these cases reference to the τ' is usually suppressed.

Our standard references in Commutative Algebra are [EIS], [MAS], and [ZS] (still!).

* * *

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I also wish to thank Professor Ernst Kunz at Regensburg for sending the book [KRD] which, along with his numerous articles, helped clarify the interconnection between various descriptions of residues.

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E.II Wiebe Duality.

Wiebe's duality results were originally given in in [WIE]. Proofs are also given in [SS] and [KKD], using determinants, their expansions, and Cramer's Rule.

For the convenience of the reader I give a more conceptual proof here in terms of free resolutions, Koszul complexes, etc. The basis is the following folk Theorem, much in the vein of the classical paper [BAS], albeit without the use of injective resolutions.

E.II.1 Theorem. *Let I, J be perfect ideals in the ring R, both of homological codimension (depth) p ($p = \text{profondeur}$). Then:*

a) $\mathrm{Hom}_R(R/I, R/J) \simeq \mathrm{Hom}_R(\mathrm{Ext}^p_R(R/J, R), \mathrm{Ext}^p_R(R/I, R))$
b) $R/J \simeq \mathrm{Ext}^p_R(\mathrm{Ext}^p_R(R/J), R)$
c) The exact annihilator of $\mathrm{Ext}^p_R(R/J, R)$ *is* J.

Proof:

a) Let $F.$ and $G.$ be projective resolutions of R/I and R/J, respectively. Consider a mapping

$$\varphi : R/I \to R/J$$

By elementary Homological Algebra, φ may be lifted to a mapping of the resolution $F.$ to the resolution $G.$. It is unique modulo homotopy. Applying the functor $^* = \mathrm{Hom}_R(\cdot, R)$ and taking cohomology, we get a uniquely defined mapping

$$\check{\varphi} : \mathrm{Ext}^p_R(R/J, R) \to \mathrm{Ext}^p_R(R/I, R)$$

The mapping

$$\mu : \varphi \to \check{\varphi}$$

is linear by uniqueness.

Now a well-known result states that the dualized complexes $\mathrm{Hom}(F., R)$ and $\mathrm{Hom}(G., R)$ are acyclic except in dimension zero, hence affording projective resolutions of the modules $\mathrm{Ext}^p_R(R/I, R)$ and $\mathrm{Ext}^p_R(R/J, R)$.

By the same argument as above, any mapping $\check{\varphi}$ from the latter module to the former induces a mapping $\nu(\check{\varphi}) = \varphi : R/I \to R/J$. By homotopy uniquness arguments, μ and ν are inverses of one another, proving statement a).

b) Again, the dual resolution $G.^*$ yields a projective resolution of $\mathrm{Ext}^p_R(R/J, R)$. Dualizing once more we retrieve the original complex $G.$, whence the result.

c) The isomorphism just established immediately shows that the annihilator J, of R/J, contains that of $\mathrm{Ext}^p_R(R/J, R)$. The reverse inclusion is obvious, whence the result.

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In the following subsections we record a number of corollaries of the above Theorem.

E.II.2 First special case.

Let $R, \mathbf{m}, k = R/\mathbf{m}$ be a regular local ring, and I an $\mathbf{m}$ -primary ideal. Using the Koszul resolution of $R/\mathbf{m}$ one proves

$$R/\mathbf{m} \simeq \mathrm{Ext}^p_R(R/\mathbf{m}, R)$$

As is well known the number of irreducible components of I equals the socle dimension of R/I, which in turn is the k-vector space dimension of the module

$$\mathrm{Hom}_R(R/\mathbf{m}, R/I)$$

By the result above, this module is isomorphic to

$$\mathrm{Hom}(\mathrm{Ext}^p_R(R/I, R), \mathrm{Ext}^p_R(R/\mathbf{m}, R)) \simeq \mathrm{Hom}(\mathrm{Ext}^p_R(R/I, R), R/\mathbf{m})$$

the dimension of which is the number of generators of the module $\mathrm{Ext}^d_R(R/I, R)$. So, of the two, the latter is cyclic if and only if the former has socle dimension 1, i.e., if and only if I is irreducible. Similarly, reversing roles, we see that

$$\mathrm{Ext}^p_R(R/I, R)$$

has socle dimension 1 (since R/I is cyclic), i.e., its zero submodule is irreducible.

E.II.3 Second special case.

Conditions as in the Theorem. Suppose more precisely that the two ideals I, J are generated by quasi-regular sequences of length p. "Quasi-regular" means that the corresponding Koszul complex is acyclic in positive dimensions. I and J are then obviously perfect.

It is well-known, and easy to prove, that

$$\mathrm{Hom}_R(R/I, R/J) \simeq (J : I)/J$$

By the Theorem, this is isomorphic to

$$\mathrm{Hom}(\mathrm{Ext}^p_R(R/J, R), \mathrm{Ext}^p_R(R/I, R)) \simeq \mathrm{Hom}(R/J, R/I) \simeq (I : J)/I$$

hence

$$(J : I)/J \simeq (I : J)/I$$

In case $I \subset J$ we get

$$R/J \simeq (I : J)/I$$

E.II.4 Third special case. Wiebe's Results

Again, I, J are assumed to be generated by quasi-regular sequences and $I \subset J$. Letting $\underline{I}$ denote a row-matrix consisting of generators of the ideal I we may write this inclusion in matrix form as

$$\underline{I} = \underline{J}A$$

where A is a p/p-matrix with elements in R.

Now the module $\mathrm{Hom}_R(R/I, R/J)$ is generated by the canonical projection $\pi : R/I \to R/J$. In the Koszul complexes associated with the generating systems above we get a mapping $\psi : R^p \to R^p$ given by the matrix A. The lifting may be chosen as the exterior powers of ψ, so, at the far end, we get the mapping $\wedge^p R^p \to \wedge^p R^p$ given by muliplication by $\det A$. From this we see that

$$(I : J)/I \simeq \mathrm{Hom}(\mathrm{Ext}^p_R(R/J, R), \mathrm{Ext}^p_R(R/I, R)) \simeq \det A \cdot R/I$$

which is isomorphic to

$$(J : I)/J \simeq R/J$$

so

$$I : J = (\det A) + I$$

which is the first of Wiebe's results.

We also see that the module

$$\mathrm{Hom}(R/J, R/I) \simeq \mathrm{Hom}(\mathrm{Ext}^p_R(R/J, R), \mathrm{Ext}^p_R(R/I, R))$$

is generated by (multiplication by) $\det A$.

So, finally,

$$I : \det A \simeq \mathrm{Ann}(\mathrm{Hom}(R/J, R/I)) \simeq \mathrm{Ann}(\mathrm{Hom}(R/I, R/J)) \simeq \mathrm{Ann}(R/J) = J$$

which is the second Wiebe Theorem.

E.II.5 Fourth special case.

The following case will be used in the section on residues, and (hopefully) in the forth-coming paper on fundamental classes. It is given as an exercise in [EIS].

Let R be a *local* Noetherian ring. Suppose $I \subset J$, I generated by a quasi-regular sequence with p elements, J perfect of homological (co)dimension p. Let $G.$ be a *free* resolution of the module R/J, of length p. Using the Koszul resolution of R/I, the projection $\pi : R/I \to R/J$ lifts to a comparison mapping, the p:th component of which is

$$\varphi_p : R \simeq \wedge^p R^p \to G_p$$

This mapping can be represented by a column matrix C.

Again, dualizing, and taking cohomology, we derive a mapping

$$\check{\varphi} : \mathrm{Ext}^p_R(R/J, R) \to \mathrm{Ext}^p_R(R/I, R) \simeq R/I$$

induced by the dual mapping

$$\varphi^* : G_p^* \to \wedge^p R^{p*}$$

Since this mapping is described by the row matrix C^t ($t =$ transpose) the image of $\check{\varphi}$ is the submodule of R/I generated by the elements of the matrix C.

The kernel of $\check{\varphi}$ is the image of

$$\mathrm{Ext}_R^{p-1}(J/I, R)$$

in the long exact sequence of Ext:s. This module is zero, since the annihilator of J/I contains a regular sequence of lenght at least p. So $\check{\varphi}$ is injective. Now, by the Theorem, the exact annihilator of $\mathrm{Ext}_R^d(R/J, R)$ is J, so the image of $\check{\varphi}$ equals $(I : J)/I$.

That is, the elements of the column matrix C generate $I : J$, modulo I.

E.III Regular Prime Bases

E.III.1 Their existence.

This section elaborates on the results in [ZS], page 310 ff. cf. also the book of Gröbner, [GRÖ].

Consider the polynomial ring $A = k[x_1, x_2, \ldots, x_n]$ and a prime ideal $\mathbf{p} \subset A$. Denote by R the quotient ring $A/\mathbf{p}$. Let $\tau_1, \tau_2 \ldots \tau_n$ denote the residue classes of $X_1, X_2, \ldots X_n$ modulo $\mathbf{p}$. Let d denote the Krull dimension of R.

By a linear change of coordinates we may assume that $\tau_1' = \tau_1, \tau_2, \ldots, \tau_d$ are algebraically independent and that the classes $\tau_{d+1}'' = \tau_{d+1}, \ldots, \tau_n$ are integral over the ring

$$k[\tau'] := k[\tau_1', \tau_2', \ldots, \tau_d']$$

So we may write

$$R = k[\tau', \tau''] = k[\tau'][\tau'']$$

We let K denote the quotient field

$$K = R_{\mathbf{p}} = k(\tau_1', \ldots, \tau_d')[\tau_{d+1}'', \ldots, \tau_n''] = L[\tau'']$$

where the field L is purely transcendental over k, and K is algebraic over L.

Consider the polynomial ring

$$L[x_{d+1}'', \ldots, x_n'']$$

The extension of $\mathbf{p}$ to this ring is zero-dimensional, hence generated by $n - d$ elements $f_{d+1}, \ldots, f_n$ which may be chosen to lie in A.

We see that the extension $\mathbf{p}A_{\mathbf{p}}$ is generated by the f_i, so the local ring $A_{\mathbf{p}}$ is regular. We say then that the f_i form a *regular prime basis* for the prime ideal $\mathbf{p}$. Choosing some suitable element g belonging to the remaining associated prime ideals of the ideal $(f) = (f_{d+1}, \ldots, f_n)$, we may write $\mathbf{p} = (f) : g$.

The following isomorphism follows readily from the above considerations:

$$K \simeq L[x_{d+1}, \ldots, x_n]/(f_{d+1}, \ldots, f_n)$$

E.III.2 Idempotents

We now regard the prime basis elements f_i as elements $f_i(\tau', x'')$ in the ring

$$K[x''] \simeq k(\tau')[\tau''] \otimes_{k(\tau')} k(\tau')[x'']$$

By simple flatness arguments they still form a quasi-regular sequence in this ring. We always think of them as

$$f_i(\tau', x'') - 0 = f_i(\tau', x'') - f_i(\tau', \tau'')$$

Obviously

$$K[x'']/(f(\tau', x'')) \simeq k(\tau')[\tau''] \otimes_{k(\tau')} k(\tau')[\tau'']$$

is reduced, since $K/k(\tau')$ is separable.

From this we see, in obvious notation, that

$$(f(\tau', x'')) = (x'' - \tau'') \cap \text{ other maximal ideals in } K[x'']$$

and, by the Chinese Remainder Theorem,

$$K[x'']/(f(\tau', x'')) \simeq K[x'']/(x'' - \tau'') \oplus ??? \quad \simeq K \oplus ???$$

We look for the idempotents belonging to this decomposition.

Let us use the same notation $(f(\tau', x''))$ and $((x'' - \tau''))$ for the column matrices made up by the respective generating systems. We then have

$$(f(\tau', x'')) = M_0(x'', \tau'')((x'' - \tau''))$$

where M_0 is a square matrix with elements in $K[x''] = k(\tau')[\tau''][x'']$. Writing $D_0(x'', \tau'')$ for $\det M_0$ we have, by Wiebe's Theorem,

$$(f(\tau', x'')) : D_0(x'', \tau'') = (x'' - \tau'')$$

$$(f(\tau', x'')) : (x'' - \tau'') = D_0(x'', \tau'')$$

proving that $D_0(x'', \tau'') \notin (f)K[x'']$, so $0 \neq D_0(\tau'', \tau'')$ in R.

By a simple division argument we may write:

$$D_0(x'', \tau'') - J = D_0(x'', \tau'') - D_0(\tau'', \tau'') = P(x'', \tau'')((x'' - \tau''))$$

since the left member vanishes on substituting $x'' \to \tau''$. P is a row matrix. So

$$1 = \frac{1}{J} P(x'', \tau'')(x'' - \tau'') + \frac{D_0(x'', \tau'')}{J} = (1 - \epsilon_0) + \epsilon_0$$

Since the terms in the right member annihilate one another modulo $(f(\tau', x''))$ their classes modulo (f) yield the desired idempotents.

E.III.3 Traces

From the above considerations we may put together the following commutative diagram

$$\begin{array}{ccc} K[x'']/(f(x',\tau'')) & \leftarrow & K[x'']/(x''-\tau'') \\ \downarrow & & \downarrow \\ k(\tau')[x'']/(f(\tau',x'')) & \rightarrow & K \end{array}$$

Here the top horizontal arrow maps 1 to an idempotent ϵ_0 displaying the right member as a direct summand of the left.

The left vertical mapping is constructed from the trace mapping

$$\mathrm{Tr}_{k(\tau')} : K = k(\tau')(\tau'') \to k(\tau')$$

and the right vertical mapping is the mapping induced by the trace mapping. It is an isomorphism.

By the commutativity of the diagram, and the isomorphism just noted, we see that

$$\mathrm{T}(D_0(x'',\tau'')) := \mathrm{Tr}\,\frac{D_0(x'',\tau'')}{J} = 1$$

Let us write

$$D_0(x'',\tau'') = \sum m_i(x'')m_i'(\tau'')$$

Here we may assume the m_i, m_i' linearly independent over $k(\tau')$, We may assume that m_1 equals 1 and that the other m_i have no constant term We then get

$$\mathrm{T}(1) = 1$$

and

$$\mathrm{T}(\text{other } m_i\text{: s}) = 0$$

By the commutativity of the diagram, and the isomorphism in the right-hand part we see:

$$T \circ \epsilon_0(m(\tau'')D_0(x'',\tau'')) = m(x'')$$

From this we immediately see that the $m_i(\tau'')$ generate K over $k(\tau')$ and that the sequences $m_i(\tau'')$, $m_i'(\tau'')$ are dual bases for the bilinear form

$$B(m,n) = T(m \cdot n)$$

from K to $k(\tau')$.

E.III.4 An extension

More generally, let $Q(f)$ denote an (f)–primary ideal in $A/\mathbf{p}$ generated by certain expressions in the $f_i(\tau',x'')$. By abuse of notation we use the same symbol for the extension to the ring $K[x'']$. Let $q(x''-\tau'')$ denote the primary component of this ideal, belonging to the maximal ideal $(x''-\tau'')$.

Again we have a commutative diagram:

$$\begin{array}{ccc} K[x'']/(Q(f)) & \leftarrow & K[x'']/(q(x''-\tau'')) \\ \downarrow & & \downarrow \\ k(\tau')[x'']/(Q(f)) & \rightarrow & K'[f]/(Q'(f)) \end{array}$$

Here again the top arrow maps 1 to an idempotent ϵ, and the left vertical arrow is constructed from the trace mapping $K \to k(\tau')$. $K' \simeq K$ denotes a Cohen field of $k(\tau')[x'']/(Q(f))$, and the bottom horizontal row comes from re-writing everything in the form $p(f) \in K'[f]$, using analytic independence of the f_i. The right vertical arrow is the composition of the three.

E.III.5 Theorem. *The composition described above is a ring isomorphism mapping the subfield $K \subset K[x'']/(q(x''-\tau''))$ isomorphically onto the Cohen field K'.*

Proof:

Let $\overline{K}$ be a normal closure of the separable extension $K/k(\tau')$. Let G denote the Galois group of $\overline{K}/k(\tau')$. For any element

$$\epsilon h \in K[x'']/(Q(f)), \quad h \in K[x]/(q(x''-\tau'')),$$

we may write, in obvious notation,

$$\mathrm{Tr}(\epsilon h) = \sum_{g \in G} \epsilon^g h^g$$

Note that the ϵ^g are orthogonal idempotents in $\overline{K}[x'']/(Q(f))$. From this we easily see

$$\mathrm{Tr}(\epsilon hk) = \sum_g \epsilon^g h^g k^g = (\sum_g \epsilon^g h^g)(\sum_g \epsilon^g k^g) = \mathrm{Tr}(\epsilon h)\,\mathrm{Tr}(\epsilon k)$$

From this description it is clear that $\mathrm{Tr} \circ \epsilon$ is injective. However, it is clear by the splitting over $\overline{K}$ that both rings have the same dimension over $k(\tau')$, namely $1/g$ times the dimension of $K[x'']/(Q(f))$, $g = [K : k(\tau')]$. So our mapping is surjective as well.

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This isomorphism is to be conceived of as a linearization of the scheme belonging to $Q(f)$. It is equivalent to a one point scheme over the function field of the underlying variety.

The inverse isomorphism $K'[f]/(Q(f)) \to K[x'']/(q(x''-\tau''))$ is given by formal Taylor expansions in powers of the $(x_i''-\tau_i'')$. This is clear since $\mathrm{Tr}(\epsilon h) = \epsilon h + \sum_{g \neq 1} \epsilon^g h^g$ maps back to h and the remaining terms belong to $q(x''-\tau'')$.

By the isomorphism just established, modulo any higher **p**-primary ideal, the ideal $q(x''-\tau'')$ is a complete intersection if $Q'(f)$ is. We will need this observation in E.IX.7.

E.III.6 Example: Let $A = k[X,Y]$ and $f(X,Y) = X^2 + Y^2 - 1$. Let us study the ring $A/(f^2)$. We let $\mu = Y + (f^2)$ be the parameter variable, and $\tau = X + (f^2)$ the fiber variable. The idempotent ϵ is then

$$\epsilon = \frac{1}{4\tau^3}[\tau(x+\tau)^2 - (x+\tau)^2(x-\tau)] = \frac{1}{2} + \frac{x}{2\tau} - \frac{x}{4\tau^3}f(x) \equiv \epsilon_0 \pmod f$$

as is easily checked. It may be constructed from the formal partial fractions decomposition

$$\text{"}\frac{1}{f^2} = \frac{1}{(x-\tau)^2(x+\tau)^2} = \frac{1}{4\tau^3}[\frac{\tau}{(x-\tau)^2} - \frac{1}{x-\tau} + \frac{1}{x+\tau} + \frac{\tau}{(x+\tau)^2}]\text{"}$$

We have

$$\mathrm{Tr}_{K/k(\mu)}(\epsilon\tau) = x + \frac{x}{2(\mu^2-1)}f = \tilde{x}$$

and it is easy to check that

$$f(\tilde{x}) \equiv 0 \pmod{(f^2)}; \qquad \tilde{x} \equiv x \pmod{(f)}$$

So we find the Cohen field

$$K' = k(\mu)[\tilde{x}]$$

In section IX. we will determine the idempotent for this decomposition as a "pre-residue". For the time being we content ourselves with the following

E.III.7 Theorem. *Consider $k(\tau')[x'']/(Q(f))$ as a subring of $K[x'']/(Q(f))$. Let*

$$K' = k(\tau')[\tilde{x}''] \simeq k(\tau')[X'']/(f(\tau', X''))$$

be a Cohen subfield of the first ring. Then the idempotent ϵ is given by

$$\epsilon = \frac{D_0(\tau'', \tilde{x}'')}{J}$$

with $J = D_0(\tau'', \tau'')$

Proof: We need only prove that $\epsilon \cdot (1-\epsilon) = 0$ and that $\epsilon - 1 \in (x'' - \tau'')$. The proof of these facts is formally the same as that in III.2., just keep in mind that all $f(\tau', \tilde{x}'') = 0$ ∎

For instance, the idempotent in E.III.6. can be written $(1+\tilde{x})/2\tau$.

E.III.8 A special choice of prime basis.

Some of our later computations will be facilitated by a more convenient choice of prime basis. For a given square matrix M, let M^* denote the *adjugate* or *cofactor* matrix of M, whose element in position (j,i) is the cofactor of the position (i,j) in M.

Let us put

$$(g(\tau', x'')) := M_0^*(x'', x'')(f(\tau', x''))$$

(g) and (f) again denoting column matrices.

Since the determinant of M_0^* is a power of the determinant of M_0, which does not belong to $\mathbf{p}$, we see that $g_{d+1}, \ldots, g_n$ form another prime basis of $\mathbf{p}$. Taking partial derivatives, w.r.t. x''_j, at (τ', τ'') (i.e., reducing modulo $\mathbf{p}$), we get:

$$\frac{\partial g}{\partial x''_j}(\tau', \tau'')) = M_0^*(\tau'', \tau'')\frac{\partial f}{\partial x''_j}(\tau', \tau''))$$

since the other term produced by the product rule vanishes at $\mathbf{p}$. Now the product column in the right member is the j:th column of the matrix $M_0^* M_0(\tau'', \tau'')$ which is

$$D(\tau'', \tau'')\begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix}$$

with the 1 in the jth row, and all other elements equal to zero.

So the g_j satisfy

$$\frac{\partial g_i}{\partial x''_j}(\tau', \tau'') = \delta_{ij}\mathcal{D}(\tau'', \tau'')$$

Stated differently, the linear part of the formal Taylor expansion of g_j, at the generic point (τ', τ''), consists of the single term $\mathcal{D}(\tau'', \tau'')(x''_j - \tau''_j)$.

E.IV Generic and True Exponentials.

E.IV.1 Their definition.

Given a Noetherian normalization

$$R = A/\mathbf{p} = k[\tau', \tau'']; \quad K = (A/\mathbf{p})_{\mathbf{p}} = k(\tau')[\tau'']$$

the *generic exponentials* belonging to the prime ideal $\mathbf{p}$ may be defined quite formally as symbols

$$G(x, \tau) = f(\tau, x'')e^{\tau' x' + \tau'' x''}$$

where $f(x'', \tau) \in K[x'']$. Here, e.g., $\tau' x'$ means $\tau_1' x_1 + \tau_2' x_2 + \ldots + \tau_d' x_d$.

The ring A acts on exponentials by differentiation:

$$h(x) * G(x, \tau) = h(\partial/\partial x)G(x, \tau)$$

Note that the parameter variables act like multiplication:

$$x_i' * f(x'', \tau)e^{\tau' x' + \tau'' x''} = \tau_i' f(x'', \tau)e^{\tau' x' + \tau'' x''}$$

whereas the action of the fiber variables is given, on the rational f-factor, by:

$$f(x'', \tau) \to \frac{\partial f}{\partial x_j''} + \tau_j'' f(x'', \tau)$$

True exponentials are defined by viewing the exponentials as formal power series, and then taking traces over the field $k(\tau')$:

$$\mathrm{Tr}_{K/k(\tau')} G(x, \tau) = g(\tau', x'')e^{\tau' x'}$$

The action of A is the same. True exponentials form a module isomorphic to the module of generic exponentials, by the non-degeneracy of the trace form.

They may be viewed as power series

$$g(\tau', x'') \in k(\tau')[[x'']]$$

multiplied by the symbol

$$e^{\tau' x'},$$

more precisely, those among them that are annihilated by some power of $\mathbf{p}$. The word "true" refers, e.g., to the fact that the parameter variables do parametrize them.

True exponentials with

$$g(\tau', x'') \in k[\tau'][[x'']]$$

may be called *regular*. We will use the same term for the corresponding $G(x, \tau)e^{\tau x}$.

True exponentials may be viewed as (exponential) generating functions for the trace form.

E.IV.2 Inverse polynomials

Exponentials may often conveniently be replaced by their formal "Laplace transforms":

$$F(\frac{1}{x''-\tau''}) \in K[\frac{1}{x''_{d+1}-\tau''_{d+1}},\dots,\frac{1}{x''_n-\tau''_n}]+ := K[\frac{1}{x''-\tau''}]\cdot \Pi_{i=d+1}^{n}\frac{1}{x''_i-\tau''_i}$$

where the plus sign signifies that every element includes all the $n-d$ fractions at least once.

The isomorphism is given by

$$(x'')^I e^{\tau x} \longleftrightarrow \frac{I\,!}{(x''-\tau'')^{I+(1)}}$$

where $I=(i_{d+1},\dots,i_n)$ is a multi-index, $(1):=(1,1,\dots,1)$, and $I\,!=\Pi_j i_j!$

In order to define an A-action compatible with this assignment it is convenient (here and later) to extend A:

$$A=k[\tau'][x'']\subset k(\tau')[\tau''][x'']=K[x'']$$

Now any element $f(\tau',x'')\in A$ may be viewed as an element of $K[x'']$, so we may use a formal Taylor expansion "at (τ',τ'')", replacing x'' by $(x''-\tau'')+\tau''$, thus re-writing f as a K-linear combination of monomials $(x''-\tau'')^J$. We then define

$$(x''-\tau'')^J\cdot\frac{1}{(x''-\tau'')^{I+(1)}}=\frac{1}{(x''-\tau'')^{I-J+(1)}}$$

if $I\geq J$, 0 otherwise. We then extend linearly.

It is clear that this action turns the above assignment into an isomorphism of A-modules.

Of course, these inverse polynomials (and power series) may be expanded in power series involving negative powers of the x'', with coefficients in K. It is essential to check that the action just defined is consistent with the usual action of polynomials on inverse power series as defined, e.g., in [NOR], i.e., the same as above, but with all $\tau''_i=0$.

The concluding Lemma below provides an essential step in our first determination of $E(A/\mathbf{p})$.

Let us look at the module

$$\mathrm{Hom}_{k[\tau']}(A,K)$$

$A=k[\tau'][x'']$ acts in the usual manner:

$$a\cdot\varphi(b):=\varphi(ab)$$

If $a = f(x')g(x'')$ this means

$$a \cdot \varphi(b) = f(\tau')\varphi(g(x'')b) \qquad (*)$$

We have an almost obvious isomorphism

$$\mathrm{Hom}_{k[\tau']}(A, K) \simeq \mathrm{Hom}_k(k[x''], K)$$

where the action of A on the right member is *defined* by the formula $(*)$ above.

Now, to every element φ in $\mathrm{Hom}_k(k[x''], K)$ we may associate an inverse power series

$$\sum \frac{\varphi(m)}{m(x'')\Pi x_i''}$$

the sum running over all monomials $m(x'') \in k[x'']$. It is easy to see that the A-action on either member commutes with this assignment. Summing up we have the following

E.IV.3 Lemma.

$$\mathrm{Hom}_{k[\tau']}(A, K) \simeq \mathrm{Hom}_k(k[x''], K) \simeq K[[\frac{1}{x''}]]_+ \simeq K[[\frac{1}{x'' - \tau''}]]_+$$

s

E.V de Rham Cohomology and Traces of Differential Forms.

E.V.1 Traces of differential forms

The standard reference for the subject of this and the next subsection is [KKD].

Deviating somewhat from our standard notation, we place ourselves in the following situation:

$$k \subset R \subset S$$

are Noetherian rings, finitely generated as k–algebras. S is a module-finite R-algebra, projective as an R-module. We denote by

$$\Omega_{S/k}$$

the module of differentials of S over k. We also introduce the notation

$$\Omega^d_{S/k} = \wedge^d \Omega_{S/k}$$

We assume further that

$$\Omega^d_{S/k} = S \otimes_R \Omega^d_{R/k}$$

for all d. The classical situation is that where R and S are fields, and S is an algebraic separable extension of R.

It is then easy to define the R-trace of a differential form $\in \Omega^d_{S/k}$. It is:

$$\mathrm{Tr}_{S/R}(g df_1 \wedge \ldots \wedge df_d) = (\mathrm{Tr}_{S/R} g) df_1 \wedge \ldots \wedge df_d$$

where $f_1, \ldots, f_d \in R$ and where the trace of $g \in S$ is defined as the trace of the R-linear mapping "multiplication by g".

It is easy to prove that the above defini tion is independent of all choices involved. Furthermore, for $g \in S$, we may write $dg = \sum_i g_i dr_i, r_i \in R, \forall i$ whence

$$\mathrm{Tr}(dg \wedge df_1 \wedge \ldots \wedge df_d) = \sum_i \mathrm{Tr}(g_i) dr_i \wedge df_1 \wedge \ldots \wedge df_d =$$

$$\mathrm{Tr}(dg) \wedge df_1 \wedge \ldots \wedge df_d$$

The proof of the following lemma is included for lack of a suitable reference:

E.V.2 Lemma. *The trace defined in the situation above commutes with exterior differentiation*

Proof: By localization arguments we may assume that S/R is *free*. We need only consider a "monomial"

$$g\omega = gdf_1 \cdot \ldots \cdot df_d; \quad f_i \in R, \forall i$$

Now

$$\mathrm{Tr}(d(g\omega)) = \mathrm{Tr}(dg \wedge \omega) = \mathrm{Tr}(dg) \wedge \omega$$

and

$$d(\mathrm{Tr}\, g\omega) = d(\mathrm{Tr}\, g) \wedge \omega$$

so it is enough to prove the Lemma for a 0-form $g \in S$.

Let $\underline{\mathbf{e}}$ denote a row matrix whose components are some basis of S/R. We may write

$$g\underline{\mathbf{e}} = \underline{\mathbf{e}}\mathcal{M}$$

$$d\underline{\mathbf{e}} = \underline{\mathbf{e}}\mathcal{N}$$

where $\mathcal{M}$ is a matrix of elements in R and $\mathcal{N}$ is a matrix of differentials. By definition,

$$\mathrm{Tr}\, g = \mathrm{Tr}\, \mathcal{M} \in R$$

and

$$d(\mathrm{Tr}\, g) = d(\mathrm{Tr}\, \mathcal{M}) = \mathrm{Tr}(d\mathcal{M})$$

where the differential of a matrix is defined element-wise. Differentiating the equation $g\underline{\mathbf{e}} = \underline{\mathbf{e}}\mathcal{M}$, we get

$$dg\underline{\mathbf{e}} + gd\underline{\mathbf{e}} = d\underline{\mathbf{e}}\mathcal{M} + \underline{\mathbf{e}}d\mathcal{M}$$

i.e.,

$$dg\underline{\mathbf{e}} + g\underline{\mathbf{e}}\mathcal{N} = d\underline{\mathbf{e}}\mathcal{M} + \underline{\mathbf{e}}d\mathcal{M}$$

i.e.,

$$dg\underline{\mathbf{e}} + \underline{\mathbf{e}}\mathcal{M}\mathcal{N} = \underline{\mathbf{e}}\mathcal{N}\mathcal{M} + \underline{\mathbf{e}}d\mathcal{M}$$

Taking traces of both members, and using $\mathrm{Tr}(\mathcal{N}\mathcal{M}) = \mathrm{Tr}(\mathcal{M}\mathcal{N})$ (note that $\mathcal{M}$ contains no differentials), we get

$$\mathrm{Tr}(dg) = \mathrm{Tr}(d\mathcal{M}) = d\,\mathrm{Tr}\,\mathcal{M} = d\,\mathrm{Tr}\, g$$

∎

E.V.3 The de Rham complex

We introduce, momentarily, indeterminates $T_1, \ldots, T_n$, and the notation $k[T] = k[T_1, \ldots, T_n]$ for another copy of the polynomial ring A. We also introduce formal exponentials $f(T,x)e^{Tx} = f(T_1, \ldots, T_n, x_1, \ldots x_n)e^{T_1x_1+\cdots+T_nx_n}$ where $f(T,x)$ is an element of $k[T,x]$

The action of $k[T]$ is by differentiation w.r.t. to the T variables, that of $k[x]$ is differentiation w.r.t to the x. These actions commute. We may replace exponentials by formal Laplace transforms $\in K[1/(x'' - T'')]_+$; the x then act by multiplication (almost) and the T by differentiation.

We may form the module M consisting of formal exponentials and the module

$$\mathcal{C}^j = M \otimes_{k[T]} \Omega^j_{k[T]/k} = M \otimes \wedge^j \Omega_{k[T]/k}$$

We turn these modules into a complex by exterior differentiation with respect to the T–variables:

$$d : \mathcal{C}^j \to \mathcal{C}^{j+1}$$

$$d(fdT) = df \wedge dT$$

We next introduce the *differential ideal* I of the exterior algebra Ω, generated by all $f, df; f \in \mathbf{p} \subset k[T]$ and form the complex

$$C. = (\mathcal{C}./I\mathcal{C}.)_{\mathbf{p}}$$

with operations induced by those of the original complex. Details of this construction are given in [KKD], p.27.ff.

We introduce the notation $\tau, d\tau$ for the residueclasses of the T, dT. The operation induced by the differentiation $\partial/\partial T_j$ will be denoted by $\partial/\partial\tau_j$.

We then have

$$C^j = M' \otimes \wedge^j \Omega_{(A/\mathbf{p})/k}$$

where M' denotes the module of formal exponentials $f(\tau, x)e^{\tau x}$ with obvious actions. Here $f(\tau, x) \in K[x]$, K still denoting the quotient field $(A/\mathbf{p})_{\mathbf{p}}$. Note that *all* variables x, not just the fiber variables x'', enter this expression. The elements of C^j look like this:

$$f(\tau, x)e^{\tau x} d\tau$$

Exterior differention in this quotient complex may be performed as ususal:

$$\begin{aligned} d(f(\tau, x)e^{\tau x} d\tau) &= d(f(\tau, x)e^{\tau x} \wedge d\tau) \\ &= \sum_{j=1}^{n} [\frac{\partial f(\tau, x)}{\partial \tau_j} e^{\tau x} \tau_j \wedge d\tau] + \sum_{j=1}^{n} [x_j f(\tau, x) e^{\tau x} d\tau_j \wedge d\tau] \end{aligned}$$

unambiguously. We get the "right" differentials, thanks to the chain rule, of course.

If $f_{d+1}, \ldots, f_n$ are a prime basis for $\mathbf{p}$, then it is well-known that

$$df_{d+1}, \ldots, df_n$$

are linearly independent, modulo $\mathbf{p}$, over the field K. From this we immediately see that

$$C^j = 0, j > d$$

Actually, referring to a Noetherian normalization $A/\mathbf{p} = k[\tau', t'']$, the "fiber differentials" $d\tau''$ are K-linear combinations of the $d\tau'$, which in turn are linearly independent.

Along with this complex we will have to consider a "traced de Rham complex", $\tilde{C}.$. We refer to a Noetherian normalization $A/\mathbf{p} = k[\tau', \tau'']$. Again we may view the expression

$$f(\tau, x)e^{\tau x}d\tau = f(\tau', t'')e^{\tau' x'}e^{\tau'' x''}d\tau \in C^j$$

as a formal power series. We also may assume that $d\tau$ involves only the parameter variables τ'.

Taking term-wise traces (from K to $k(\tau')$) we obtain expressions

$$g(\tau', x)e^{\tau' x'}d\tau'$$

which are annihilated by some power of $\mathbf{p}$. They form a module $\tilde{C}^j$ isomorphic to C^j. Since the trace commutes with exterior differentiation, the complex $\tilde{C}.$ is isomorphic to $C.$, hence has the same cohomology.

E.VI First Determination of $E(A/\mathbf{p})$

Our first determination of the injective envelope of the module $A/\mathbf{p}$ will depend on a normalization. The next section offers an invariant description.

We start off with the following Lemma, the proof of which may be found, e.g., in [HAR], p. 213:

E.VI.1 Lemma. *Let E be an injective A-module, A Noetherian. Let $\mathbf{p} \in A$ be a prime ideal. Let $E_0 = \Gamma_{\mathbf{p}}(E)$ be the the submodule of E consisting of elements annihilated by some power of $\mathbf{p}$. Then E_0 is itself an injective A-module.*

▮

Actually, $\mathbf{p}$ may be replaced by *any* ideal.

We wish to prove that the module of generic exponentials, or, equivalently, the module of inverse polynomials

$$E_1 = K[\frac{1}{x'' - \tau''}]_+$$

is an (the) injective envelope of the A-module $A/\mathbf{p}$. There are two things to prove: that E_1 is an essential extension of K, and that E_1 is injective. Our next lemma takes care of the first property.

E.VI.2 Lemma. *The module*

$$K[\frac{1}{x'' - \tau''}]$$

of inverse polynomials is an essential extension of K. Also, each element is annihilated by some power of the prime ideal $\mathbf{p}$.

Proof: Consider any element

$$F(\frac{1}{x'' - \tau''}) = \sum_I a_I \frac{1}{(x'' - \tau'')^{I+(1)}}; \quad a_I \in K$$

The multi-indices involved may be equipped with an obvious partial order. Let $J + (1)$ be maximal with respect to this order. Let $g_{d+1}, g_{d+2}, \dots g_n$ be the special prime basis introduced in III.7. Let $g := \Pi_k g_k^{j_k}$. Using a formal Taylor expansion "at τ''", and the fact that

$$\frac{\partial g_i}{\partial x''_j}(\tau', \tau'') = \delta_{ij} \cdot \mathcal{D}(\tau'', \tau'')$$

one easily sees that

$$0 \neq g \cdot F \in K$$

For the last statement, note that the g_j generate the extension of the prime ideal $\mathbf{p}$ to the localization $A_{\mathbf{p}}$. Since the inverse polynomials have coefficients in K it is enough to prove that a large enough power g^k of the ideal (g_j) annihilates the element F. But this is easy; simply take k so large that any monomial in the g_j has one exponent greater than the corresponding one in I.

▮

As a preliminary step towards the second property we exhibit the inverse polynomials as submodule of a much larger injective module:

E.VI.3 Lemma. *The module*

$$K[[\frac{1}{x'' - \tau''}]]+$$

of inverse power series is an injective A-module

Proof: We first note the well-known fact that the quotient field $k(\tau')$ is an injective module over the ring $k[\tau']$. So is K, being a finite-dimensional vector space over $k(\tau')$.

By a standard change-of-rings argument, cf. any standard text on Homological Algebra or the book [VAS], the module

$$\mathrm{Hom}_{k[\tau']}(A, K)$$

is then an injective A-module. However, in IV.3. we noted the isomorphism

$$\mathrm{Hom}_{k[\tau']}(A, K) \simeq K[[\frac{1}{x'' - \tau''}]]+$$

▮

We finally turn to the proof of the main result of this section. Note that we are proving the structure of injective envelopes without presupposing their existence. With a judicious choice of definitions and categories we might even do away with the reliance on Zorn's lemma, and similar black magic, altogether!

E.VI.4 Theorem. *The module of inverse polynomials*

$$K[\frac{1}{x'' - \tau''}]+$$

is an injective envelope of the A-module $A/\mathbf{p}$

Proof: By the lemma above all we need to do is exhibit this module as a direct summand of the $\mathbf{p}$-primary part E_0 of the module of inverse power series. Let (g) be the prime basis discussed previously (Section III.7.). Obviously, E_0 is the increasing union of the submodules M_k annihilated by $(g)^k$ and it is enough to show that the submodule (of M_k) consisting of inverse polynomials is a direct summand of M_k.

Again we use the embedding

$$A = k[\tau'][x''] \subset K[x''] = k(\tau')[\tau''][x'']$$

and the obvious action of the latter ring on M_k described in IV.2. (this is actually a substitute for completing the local ring $A_{\mathbf{p}}$). If we can exhibit the module of inverse polynomials as a $K[x'']$-direct summand of M_k we are done.

However, by the discussion in III.2. we may write

$$(g(\tau', x'')) = (x'' - \tau'') \cap \text{ other maximal ideals of } K[x'']$$

From this, and the special choice of the g_i, we see:

$$(g)^k = ((x'' - \tau''))^k \cap I$$

where I is the intersection of primary ideals belonging to those other maximal ideals.

By the Chinese Remainder Theorem, M_k is then the direct sum of the submodule killed by $((x'' - \tau''))^k$ and that killed by I. The first, however, is obviously the submodule of inverse polynomials in the $x'' - \tau''$. ∎

.

Remark: It can be proved that E_0 is actually the direct sum of g copies of $E(A/\mathbf{p})$, where $g = [K : k(\tau')]$, the *degree* of the variety $V(\mathbf{p})$ belonging to the prime ideal $\mathbf{p}$. On taking traces, with respect to $k(\tau')$, all of them are mapped isomorphically onto the $\mathbf{p}$-primary part of the injective module

$$\mathrm{Hom}_{k[\tau']}(A, k(\tau')) \simeq \mathrm{Hom}_{k(\tau')}(k[\tau', x''], k(\tau')) \simeq$$

$$\simeq \mathrm{Hom}_k(k[x''], k(\tau')) \simeq k(\tau')[[\frac{1}{x''}]]_+$$

which may be identified with the module of *true* exponentials.

Remark: In III.4. we noted the K-isomorphism

$$K[f]/(Q'(f)) \simeq K[x'']/(q(x'' - \tau''))$$

where $(Q'(f))$ and $(q(x'' - \tau''))$ are primary ideals.

By cofinality we must have

$$\lim_{\rightarrow} \mathrm{Hom}_K(K[f]/(f)^{(d)}, K) \simeq \lim_{\rightarrow} \mathrm{Hom}_K(K[x'' - \tau'']/((x'' - \tau''))^{(d)}, K)$$

i.e.,

$$K[\frac{1}{f}]_+ \simeq K[\frac{1}{x'' - \tau''}]_+$$

which gives us another description of the injective envelope $E(A/\mathbf{p})$.

Actually, if $(g) \subset \mathbf{p}$ is *any* system of parameters of $A_{\mathbf{p}}$ one may prove, much in the same manner, that

$$E(A/\mathbf{p}) \simeq k(\tau')[x'']/(g)[\frac{1}{g}]_+$$

E.VII Second Determination of $E(A/\mathbf{p})$

In this section we prove the following theorem, thus finally giving an invariant description of injective envelopes, i.e., one independent of the normalization.

E.VII.1 Theorem. *The injective envelope $E(A/\mathbf{p})$ is isomorphic to the highest cohomology module of the de Rham complex described in V.3.*

The proof will be given in several steps. The easy part is provided by the following Lemma.

E.VII.2 Lemma. *Let $\mathbf{p}$ be a d-dimensional prime ideal of A. Let $A = k[\tau', \tau'']$ be a Noetherian normalization. Then every cohomology class of $H^d(C.)$ contains an element of the form*

$$g(\tau, x'')e^{\tau x} d\tau'$$

i.e., one involving the fiber variables x'' only.

Proof: As $C^{d+1} = 0$ any element of C^d is a cocycle. Consider an element of the following form:

$$\omega = (x'_j)^k f(\tau, x', x'')e^{\tau x} d\tau'_1 \wedge \ldots \wedge d\tau'_d$$

Here we assume that x'_j is missing from f. Obviously any element of C^d is a K-linear combination of elements of this kind. It is enough to prove that this form is cohomologous to some other form of lower degree in x'_j, and at most the same degree in the remaining x'. A simple induction will do the rest.

Let $d\tau'_{[j]}$ denote the form

$$(-1)^{j+1} d\tau'_1 \wedge \ldots \widehat{d\tau'_j} \ldots \wedge d\tau'_d$$

(where the hat means "omit"), and $d\tau' = d\tau'_1 \wedge \ldots \wedge d\tau'_d$ Then

$$d[(x'_j)^{k-1} f(\tau', x', x'')e^{\tau x} d\tau'_{[j]}] =$$

$$(x'_j)^k f(\tau', x', x'')e^{\tau x} d\tau' + (x'_j)^{k-1} e^{\tau x} df(\tau', x', x'') \wedge d\tau'_{[j]}$$

The first term is the one under consideration. The second term contains x'_j to a lower degree. Differention of f, w.r.t. to the τ-variables, will not affect the degrees of the remaining x:s. The differential factor arising this way may be re-arranged into one containing only the $d\tau'$:s, using the dependence of the $d\tau''$ on the $d\tau'$. ∎

Next we have to show that the class of any

$$f(\tau, x'')e^{\tau x} d\tau'$$

is non-zero. This will establish the isomorphism claimed in the theorem. We will prove the case $\mathbf{p} = 0$ in detail, and then sketch the reduction from the general case to this case. Note that in this case all the variables x_j are parameter variables. So what we need is the following:

E.VII.3 Lemma. *Suppose* $\mathbf{p} = 0$*, the zero-ideal in* A*. Suppose further*

$$d\sum_{i=1}^{d} g_i(\tau, x)e^{\tau x}d\tau'_{[i]} = f(\tau)e^{\tau x}d\tau'$$

Then $f(\tau) = 0$.

Proof: We are assuming

$$\sum_i \left(\frac{\partial g_i}{\partial \tau_i} + x_i g_i\right) = f(\tau)$$

Our proof will proceed by induction on the highest degree of the g_i, in the x. We donte that degree by m. The case where all g_i are independent of the x_j is obvious.

We introduce the following notation. (g_i) is a column matrix, with the $g_i, i = 1, \ldots d$ as elements. We may write

$$(g_i) = \sum_{k=0}^{m} \overline{g}_k$$

where the k–term is a column containing the k–degree parts of the g_i. We introduce the two operators

$$\mathrm{D}; \quad \mathrm{D}(g_i) = \sum g_i x_i d\tau'_i$$

$$\delta; \quad \delta(g_i) = \sum \frac{\partial g_i}{\partial \tau'_i} d\tau'_i$$

It is easy to check the properties

$$\delta^2 = \mathrm{D}^2 = \delta\,\mathrm{D} + \mathrm{D}\,\delta = 0; \qquad \mathrm{d} = \delta + \mathrm{D}$$

We may think of D as a Koszul differential and δ as belonging to an "ordinary" deRham complex, "without exponential factors".

Our assumption may now be written

$$\begin{aligned} \mathrm{D}\,\overline{g}_m &= 0 \\ \mathrm{D}\,\overline{g}_{m-1} + \delta\overline{g}_m &= 0 \\ \mathrm{D}\,\overline{g}_{m-2} + \delta\overline{g}_{m-1} &= 0 \\ \ldots &= \ldots \\ \delta\overline{g}_0 &= f \end{aligned}$$

From the first equation, and the exactness of an appropriate Koszul complex we find a matrix $\overline{h}_{m-1}$ satisfying

$$\overline{g}_m = \mathrm{D}\,\overline{h}_{m-1}$$

Let us set

$$\overline{g}'_{m-1} = \overline{g}_{m-1} - \delta\overline{h}_{m-1}$$

By the above considerations we get

$$\begin{aligned} \mathrm{D}\,\overline{g}'_{m-1} &= 0 \\ \mathrm{D}\,\overline{g}_{m-2} + \delta\overline{g}'_{m-1} &= 0 \\ \ldots &= \ldots \\ \delta\overline{g}_0 &= f \end{aligned}$$

which is of the same form as the previous set of identitites, but of degree $m-1$ instead of m. So induction on the total degree m yields the desired conclusion $f = 0$ ∎

*R*emark: From the computations of the proof it is easy to see that the next highest cohomology is zero. We only needed the exactness of a Koszul complex in dimension 1, and the commutativity of derivations. By the same technique we could prove that our deRham complex is exact in all dimensions except the highest.

So now we have proved the isomorphism of the theorem for the n-dimensional zero ideal. Let us now deal briefly with the general situation, i.e., that of an arbitrary d-dimensional prime ideal $\mathbf{p}$.

Suppose we have

$$d[\sum_i g_i(\tau, x)e^{\tau x}d\tau'_{[i]}] = f(\tau, x'')e^{\tau x}d\tau'$$

where the right member involves the fiber variables x'' only. We must prove that $f = 0$.

Passing to the isomorphic "traced" de Rham complex, and noting that tracing commutes with exterior differentiation, we get something like this:

$$d[\sum_i \tilde{g}_i(\tau', x)e^{\tau' x'}d\tau'_{[i]}] = \tilde{f}(\tau', x'')e^{\tau' x'}d\tau'$$

Since exterior differentiation in this case involves only the parameter variables τ' we may separate this equation into its components according to various momomials in the x'';

$$\tilde{g}_i(\tau, x) = \sum_{i,I} \tilde{g}_{i,I}(\tau', x')(x'')^I,$$

$$\tilde{f}(\tau', x'') = \sum_I \tilde{f}_I(\tau')(x'')^I,$$

$$d\sum_i \tilde{g}_{i,I}(\tau', x')d\tau'_{[i]} = \tilde{f}_I(\tau')d\tau'$$

We are then back in the previous situation (over the polynomial ring $k[\tau'_1, \ldots, \tau'_d]$.) So, in $\tilde{f} = \sum_I \tilde{f}_I(\tau')(x'')^I$ we may conclude that each component, hence all of $\tilde{f}$, hence f itself, equals zero.

This concludes the proof of the Theorem.

E.VIII Noetherian Operators. Reciprocity. Duality.

To a generic exponential

$$f(\tau, x'')e^{\tau' x' + \tau'' x''} = \sum g_i(\tau) m_i(x'') e^{\tau' x' + \tau'' x''}$$

we may associate a *Noetherian operator*

$$f(\tau, \frac{\partial}{\partial x''}) = \sum_i g_i(\tau) m_i(\frac{\partial}{\partial x''})$$

where the m_i are various monomials.

These are viewed as acting "at (τ', τ'')". i.e. first they act, then everything is reduced modulo $\mathbf{p}$ (substitute $x \to \tau$). Noetherian operators are $k(\tau')$-linear.

If

$$f(\tau, x'')e^{\tau' x' + \tau'' x''} \longleftrightarrow f(\tau, \frac{\partial}{\partial x''})$$

then

$$\frac{\partial f}{\partial x''_i} e^{\tau x} \longleftrightarrow [f(\tau', \frac{\partial}{\partial x''}), x''_i]$$

the commutator of a differential operator and a multiplication..

By Taylor's Formula,

$$[f(\tau, \frac{\partial}{\partial x''}) g(x', x'')]_{x=\tau} = 0$$

$$\Longleftrightarrow$$

$$[g(\tau', \frac{\partial}{\partial x''} + \tau'') f(\tau, x'')]_{x''=0} = 0$$

$$\Longleftrightarrow$$

$$g(\frac{\partial}{\partial x})[f(\tau, x'')e^{\tau x}]_{x''=0} = 0$$

It is enough to check monomials.

From this one easily concludes

E.VIII.1 Theorem. "Reciprocity Theorem". *Let $I = (q_i)$ be a $\mathbf{p}$-primary ideal. Then the ideal I, acting in the usual manner, kills the generic exponential $f(\tau, x'')e^{\tau x}$ if and only the Noetherian operator $f(\tau, \frac{\partial}{\partial x''})$, along with all its commutators $[f(\tau, \frac{\partial}{\partial x''}), g(x', x'')]$, kills all the q_i at the generic point (τ', τ'')..*

Proof: First of all, for any $q = q_i$

$$f(\tau, \frac{\partial}{\partial x''})(g(x', x'')q(x', x''))(\tau) = 0$$

is the same as

$$(g(\tau', \tau'')f(\tau, \frac{\partial}{\partial x''})q + [f(\tau, \frac{\partial}{\partial x''}), g(x', x'')]q)(\tau) = 0$$

i.e., f kills the ideal $\mathbf{q}$ (at $\mathbf{p}$ if and only if f, and all its commutators, kill the generators q_i.

From the above equivalences, and the correspondence "derivative $\leftrightarrow$ commutator", we see that this happens if and only if the q_i kill the exponential $f(\tau, x'')e^{\tau x}$, along with all its derivatives, at $x'' = 0$. However, by Taylor's Formula, this is the same thing as the the q_i, hence also the ideal $\mathbf{q}$, killing $f(\tau, x'')e^{\tau x}$ altogether.

∎

Finding those Noetherian operators that characterize I is therefore the same as finding the generic exponential solutions $f(\tau, x'')e^{\tau x}$ of the system

$$q_i(\frac{\partial}{\partial x})(f(\tau, x'')e^{\tau x}) = 0, \forall i$$

These may also be characterized as the module

$$\mathrm{Hom}(A/I, E(A/\mathbf{p}))$$

denoted by $(A/I)^*$.

Proving that these exponential solutions characterize I amounts to proving that the canonical mapping

$$A/I \longrightarrow (A/I)^{**}$$

is *injective*.. i.e, to each $0 \neq a \in A/I$ there exists a mapping $\varphi \in (A/I)^*$, with $\varphi(a) \neq 0$.

The proof of this proceeds exactly as the corresponding proof in [MAS], p. 148, the first step in Matlis Duality.

Consider a as an element of the local ring $B = (A/I)_{\mathbf{p}}$. Then the composition

$$Ba \simeq B/\ \mathrm{Ann}\ a \to B/B\mathbf{p} = (A/\mathbf{p})_{\mathbf{p}} \to E(A/\mathbf{p})$$

is non-zero since the second mapping is surjective and the last injective. By the definition of injective modules, this mapping extends to a mapping $B \to E(A/\mathbf{p})$.

Alternatively one proceeds by induction on the $\mathbf{p}$-length of A/I, the case $A/\mathbf{p}$ being obvious. Suppose $I \subset J \subset \mathbf{p}$, with $(J/I)_{\mathbf{p}} \simeq (A/\mathbf{p})_{\mathbf{p}}$, hence A/J of $\mathbf{p}$-length one less than A/I.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A/\mathbf{p} & \to & A/I & \to & A/J & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (A/\mathbf{p})^{**} & \to & (A/I)^{**} & \to & (A/J)^{**} & \to & 0 \end{array}$$

Here both lines are exact, by exactness of the $*$ functor. The first vertical arrow is obviously injective, the rightmost arrow is injective by the induction assumption. From this it follows, by elementary Homological Algebra, that the middle vertical arrow is injective.

Of course, on localizing at $\mathbf{p}$ all injections turn into isomorphisms, and the $\mathbf{p}$-length of $(A/I)^*_{\mathbf{p}}$ equals that of $(A/I)_{\mathbf{p}}$ etc.

We record the following Duality Theorem:

E.VIII.2 Theorem. *Let $m_j(\tau', x''), j = 1, 2, \ldots m$ be a K-basis of*

$$k(\tau')[x'']/(Q(f(\tau', x''))) \simeq K[x'']/(q(x'' - \tau''))$$

Then there are Noetherian operators

$$m_j^*(\tau, \frac{\partial}{\partial x''})$$

and corresponding exponentials

$$m_j^*(\tau, x'')e^{\tau x}$$

such that

$$m_j^*(\frac{\partial}{\partial x''})m_i(\tau', x'')_{x=\tau} = \delta_{ij}$$

or, equivalently,

$$m_i(\frac{\partial}{\partial x})(m_j^*(\tau, x'')e^{tx})_{x=0} = \delta_{ij}$$

Proof: . This is an immediate consequence of our previous identifications and ordinary zero-dimensional Matlis Duality over the field K. ∎

Remark: On localizing, and completing the ring A, at $\mathbf{p}$, it turns out that the Noetherian operators are not only linear over $k(\tau')$, but linear over K as well, where (by abuse of notation) K denotes the Cohen field (field of representatives) containing $k(\tau')$.

s

E.IX Residues and Noetherian Operators (Sketch).

E.IX.1 First discussion, relying on a normalization

Here we sketch the connection between Noetherian operators and residues, quite informally. The reader will have to connect this further with his/her favorite residue. See, e.g., [BEA], [KKD], [KRD], [KCM], [HOP], [SS].

We remind the reader that we ignore all questions of regularity, i.e., we do not care whether our constructions land in a certain ring or some ring of fractions of it. We would need fundamental classes for these questions. The ring A may therefore often conveniently be replaced by its localization $S = A_{\mathbf{p}}$.

In section IV.4. (Remark) above we noted the A–isomorphism between two descriptions of $E(A/\mathbf{p})$:

$$K[1/f]_+ \simeq K[\frac{1}{x'' - \tau''}]_+$$

where the $f_i \in \mathbf{p}$ are a regular prime basis. The left member is an A-module by its construction as direct limit of Hom modules; actually it is even easier to describe the action of $\widehat{A}_{\mathbf{p}} = K[[f]]$ (Cohen Structure Theorem; easily deduced, in this special case, from III.5.)

Recall that inverse polynomials are conceived of as certain linear forms from a polynomial ring to the field K

Appending differential symbols we may normalize thus:

$$K[1/f]dT' \wedge \frac{df}{\Pi_{j=d+1}^{n} f_j} \to K[\frac{1}{x'' - t''}]\frac{d\tau'}{\Pi(x'' - \tau'')}$$

with

$$dT' \wedge \frac{df}{f} = dT' \wedge \frac{df}{\Pi_j f_j} \to \frac{d\tau'}{\Pi(x'' - \tau'')}$$

both members signifying linear forms sending 1 to 1, and monomials of positive degree to zero.

Here the differential in the left member is viewed as belonging to

$$\Omega^n_{R/k}; \qquad R = k[T_1, \ldots, T_n]_{\mathbf{p}}$$

and that of the right member to

$$\Omega^d_{k[\tau']/k}$$

The *pre-residue* is defined as the above isomorphism followed by taking the coefficient of

$$\frac{d\tau'}{\Pi_j(x''_j - \tau''_j)}$$

yielding a K-linear mapping

$$K[\frac{1}{f}]_+ \otimes \Omega^n \longrightarrow K[\frac{1}{x'' - \tau''}]_+ \otimes \Omega^d \to K$$

Viewing inverse polynomials as exponentials, this is the same as evaluation at $x'' = 0$.

The *Grothendieck residue* follows this by the trace mapping $K \to k(\tau')$.

It is fairly easy to prove that any element of the following type is mapped to zero under the pre-residue map:

$$dT' \wedge \frac{\alpha df}{f_{d+1}^{i_{d+1}} \cdot \ldots \cdot f_j^{i_j} \cdot \ldots \cdot f_n^{i_n}},$$

where at least one $i_j \geq 2$

Simply note that such a form signifies a linear form sending 1 to zero.

This means that the preresidue of any exact form of the following type:

$$d(\frac{dT' \wedge \gamma}{f^I}), \quad I \geq (1),$$

is zero

Now, for any $\mathbf{p}$-primary ideal $\mathbf{q}$, there is some k such that $(f)^{(k)} \subset \mathbf{q}$. There is then a mapping

$$\mathrm{Ext}_S^{n-d}(S/S\mathbf{q}, S) \to \mathrm{Ext}_S^{n-d}(S/(f)^{(k)}, S)$$

induced by the canonical projection $\pi : R/(f)^{(k)} \to R/\mathbf{q}$. It depends on the choice of a minimal generating system but is uniquely determined up to a unit factor (in $S = A_{\mathbf{p}}$.)

It may be constructed in the following manner. $S = A_{\mathbf{p}}$ is a regular local ring, of dimension $n - d$. The modules $A/\mathbf{q} \otimes S$, $A/(f)^k \otimes S$ admit free resolutions $G.$ and $F.$ of length $n - d$. Here $F.$ may be chosen to be a Koszul complex, with F_{n-d} of rank 1 and

$$\mathrm{Ext}_S^{n-d}(S/(f)^{(k)}, S) \simeq S/(f)^{(k)}$$

The mapping $G_{n-d} \to F_{n-d}$ is described by a row matrix (with elements in $\mathbf{p}S$), as explained in II.5. and this row matrix induces an injection

$$\mathrm{Ext}_S^{n-d}(S/\mathbf{q}S, S) \to \mathrm{Ext}_S^{n-d}(S/(f)^{(k)}, S) \simeq S/(f)^{(k)}$$

mapping the first module isomorphically onto $(f)^{(k)} : \mathbf{q}S/(f)^{(k)}$.

Now it is well-known that an element of

$$\mathrm{Ext}_S^{n-d}(S/(f)^{(k)}S, S) \otimes \Omega^n \simeq \mathrm{Ext}_S^{n-d}(S/(f)^{(k)}, \Omega^n/(f)^{(k)}\Omega^n)$$

or, more generally,

$$\mathrm{Ext}_S^{n-d}(S/(f)^I S, \Omega^n / f^{(I)} \Omega^n)$$

may be represented by a *residue symbol*

$$\begin{bmatrix} h(T)dT \\ f_{d+1}^{i_{d+1}} \cdot \ldots \cdot f_n^{i_n} \end{bmatrix}$$

or

$$\begin{bmatrix} h(T)dT \\ \ldots f_m^{i_m} \ldots \end{bmatrix}$$

having all the right properties.

One uses the dual of the Koszul complex belonging to $(f)^I$ to determine the Ext-module. It ends with $0 \leftarrow S^* \leftarrow S^{(n-d)*}$. The symbol corresponds to the class of the element mapping 1 to $h(T)dT$.

Here $h(T) \in S$. It may, however, also be viewed as an element of

$$S/(f)^I S = k(T')[T'']/(f)^I$$

which we may in turn write as $K[f]/(f)^I$, according to III.4.-III.5. K is a Cohen subfield, previously notated K'.

By this token we get mappings

$$\mathrm{Ext}_S^{n-d}(S/(f)^{(k)}, S) \otimes \Omega^n \to K[\frac{1}{f}]_+ \otimes \Omega^n \to K[\frac{1}{x'' - T''}]_+ \otimes \Omega^d \to K$$

We may view $\mathrm{Ext}_S^{n-d}(S/(f^I), S)$, $I \leq J$ as a submodule of $\mathrm{Ext}_S^{n-d}(S/(f^J; S)$. The mapping is the multiplication by the determinant f^{J-I} induced by comparison of Koszul complexes. From this one sees that the obvious mapping

$$\lim_{\rightarrow} \mathrm{Ext}_S^{n-d}(S/f^J, S) \otimes \Omega^n \to K[1/f] \otimes \Omega^n$$

is well defined. It is an isomorphism.

Now, we may view the residue symbols as differential forms with denominators, just as in the module $K[1/f] \otimes \Omega^n$ (as long as we keep fixed the order of the factors in the denominator). Then any exact differential form (where we revert to differentiations w.r.t. T', T'') will map to a exact form (and vice versa) on re-writing everything as polynomials in T', f, with coefficients in K (modulo $(f)^I$):

$$\begin{bmatrix} a(T', T'')dT' \wedge df \\ f^I \end{bmatrix} = \begin{bmatrix} p(T, f)dT' \wedge df \\ f^I \end{bmatrix} \to \frac{p(T, f)dT' \wedge df}{f^{(I)}}$$

To see this we need only convince ourselves that elements $\alpha \in K$ may be treated as differential constants $\pmod{f^I, dT'}$, for any $I \geq (1)$ But from the polynomial equations they satisfy $\pmod{f^J}$ over $k(T')$ it is clear that

$$d\alpha \equiv 0 \pmod{dT', f^J, f^{J-(1)}df}.$$

(Of course, it is more comfortable pass to the completion $\widehat{A}_{\mathbf{p}} = K[[f]]$; the subfield K is then differentially trivial over $k(\tau')$, and we need not reduce modulo anything.) This fact is usually stated as the *formula of exterior differentiatation*:

$$\begin{bmatrix} d\omega \\ f_{d+1}^{k_{d+1}} \cdots \cdot f_n^{k_n} \end{bmatrix} = \sum_j k_j \begin{bmatrix} df_j \wedge \omega \\ f_{d+1}^{k_{d+1}} \cdots f_j^{k_j+1} \cdots f_n^{k_n} \end{bmatrix}$$

In IX.3. we will apply this only to forms ω of the type $\omega = dT' \wedge \gamma$. We may then intepret the formula as both members having the same pre-residue.

E.IX.2 Example:

Just to give a hint of what is involved in IX.1. we give this tiny example Let $A = k[x, y] = k[T, U]$ and $f = x^2 + y^2 - 1$. Let us look at

$$"d \begin{bmatrix} TdU \\ f \end{bmatrix} = \begin{bmatrix} dTdU \\ f \end{bmatrix} - \begin{bmatrix} TdfdU \\ f^2 \end{bmatrix} = \begin{bmatrix} (T - f/2T)dUdf \\ f^2 \end{bmatrix}"$$

Here, $\widetilde{T} = T - f/2T \equiv T \pmod{f}$, and $f(\widetilde{T}) \equiv 0 \pmod{f^2}$, so $\widetilde{T}$ generates the Cohen field K, over $k(\tau')$, and the symbol maps to

$$\frac{\widetilde{T}dUdf}{f^2} = d\frac{+\widetilde{T}dU}{f}$$

E.IX.3 The pre-residue as a Noetherian operator

Our identifications make it clear that the preresidue

$$\Phi(a(T), h(T)) = \operatorname{Res} \begin{bmatrix} a(T)h(T)dT \\ (f)^I \end{bmatrix}$$

equals zero for all $a(T) \in A_{\mathbf{p}}$ if, and only if, $h \in (f)^I$. In other words, the K–bilinear form Φ is non-degenerate modulo $f^I(T)$. So, if the row matrix mentioned above is $C = \begin{pmatrix} d_1 & d_2 & \ldots & d_s \end{pmatrix}$, then h belongs to $\mathbf{q}$ if, and only if, the preresidues of all

$$\begin{bmatrix} a(T)d_i(T)h(T)dT \\ (f)^I \end{bmatrix}$$

are zero.

It remains to prove that the preresidue is a Noetherian operator.

This is most conveniently done by inserting a generic factor e^{Tx} (taking care of the arbitrary factor $a(T)$) in the numerator of the residue symbol.

By the exterior differentiation formula and an easy partial integration argument never involving the T'–variables (the reader may wish to consult [AIY]) one proves that the symbol

$$\begin{bmatrix} h(T)e^{Tx}dT \\ (f)^{(k)} \end{bmatrix} = \begin{bmatrix} h(T)/J\ e^{Tx}dT' \wedge df \\ (f)^{(k)} \end{bmatrix}$$

(where the Jacobian

$$J = \frac{d(f_{d+1}, \ldots, df_n)}{d(x''_{d+1}, \ldots, x''_n)} \notin \mathbf{p}),$$

may be replaced by a symbol

$$\begin{bmatrix} k(T, x'')e^{Tx}dT' \wedge df \\ f_{d+1} \cdots f_n \end{bmatrix}$$

the preresidue of which is

$$k(\tau, x'')e^{\tau x}$$

So the preresidue was obtained by applying the Noetherian operator

$$k(\tau, \frac{\partial}{\partial T''})$$

corresponding to this exponential. This operator then generates the Noetherian operators of $f^I S$, i.e., $f^I S$ is characterized by k and its commutators.

Finally, we see that

$$k(\tau, \frac{\partial}{\partial T''})(d_i\mathbf{q})(\tau) = 0$$

$$\Longleftrightarrow$$

$$[k, d_i] * (\mathbf{q})(\tau) + d_i(\tau)(k * (\mathbf{q}))(\tau) = 0$$

$$\Longleftrightarrow$$

$$[k, d_i] * (\mathbf{q})(\tau) = 0$$

the $*$ denoting action by differentiation. Note that the $d_i \in \mathbf{p}S$, so $d_i(\tau) = 0$.

We record this as the following Theorem:

E.IX.4 Theorem. *If $k(\tau, \partial/\partial x'')$ is a generating Noetherian operator for the ideal f^I (or, more generally, for a locally complete intersection $(g_{d+1}, \ldots, g_n)$), and $\mathbf{q}$ is a $\mathbf{p}$–primary ideal with $f^I : \mathbf{q} = (d_1, \ldots, d_s)$, (or $(g) : \mathbf{q} = (d)$), then the commutators $[k(\tau, \partial/\partial x''), d_i(\tau', x'')]$ are generating Noetherian operators for the ideal $\mathbf{q}$.*

▮

Stated differently, if the exponential

$$k(\tau, x'')e^{\tau x}$$

generates the solution space for the system

$$f_j^{i_j} * p(\tau, x'')e^{\tau x} = 0; \quad j = d+1, \ldots, n$$

then the derivatives

$$[d_i(\frac{\partial}{\partial x})k(\tau, x'')]e^{\tau x}$$

generate the solution space of the system

$$\mathbf{q} * p(\tau, x'')e^{\tau x} = 0$$

This is particularly nice in case $\mathbf{q}$ is irreducible, since there is then only one d_i.

E.IX.5 The pre-residue as a cohomology class ("Leray residue").

In the last subsection we constructed the Noetherian operators as the pre-residues

$$\lim_{\rightarrow} \mathrm{Ext}_S^{n-d}(S/f^I S, S) \otimes \Omega^n \to K[\frac{1}{f}]_+ \otimes \Omega^n \to K[\frac{1}{x'' - \tau''}]_+ \otimes \Omega^d \to K$$

where the symbol Ω^d refers to $\Omega^d_{k[T']/k}$.

It is natural to view the third member, the module of generic exponentials, as a submodule of

$$K[\frac{1}{x - \tau}]_+ \otimes \overline{\Omega}^d$$

where

$$\overline{\Omega}^d = \Omega^d_{k[T', T'']/k}/I$$

and I is the differential ideal generated by all $p, dp; p \in \mathbf{p}$. Here *all* T–variables enter the construction of inverse polynomials, or formal exponentials.

(if we want to stress the latter point of view the mapping

$$K[1/f]_+ \otimes \Omega^n \to K[\frac{1}{x - \tau}]_+ \otimes \overline{\Omega}^d$$

could be written

$$\frac{G(T, x)e^{Tx}}{f^I} dT \wedge \frac{df}{f} \to g(\tau, x)e^{\tau x} d\tau$$

which looks like some kind of Laplace inversion. The A–action on both members is differentiation w.r.t. to the x variables.)

$\overline{\Omega}^d$ is just our old C_d in the de Rham complex constructed in V.3. We wish to show that our pre-residue may unambiguously be defined as a cohomology class in $H^d(C.)$.

The point is this. We have been relying on a normalization $k[T', T'']$. Suppose we use some other normalization $k[U', U'']$, constructing thereby a mapping in

$$K[\frac{1}{f}]_+ dU' \wedge df \to K[\frac{1}{x'' - \mu''}]_+ d\mu'$$

Now any element in the left member also admits the description

$$p(\frac{1}{f}) dT' \wedge df$$

so we have two constructions.

However, in either case, the scheme is

$$\text{given form } \omega \sim \omega_0 \wedge \frac{df}{f} \to \overline{\omega}_0$$

with, e.g., $\omega_0 = \alpha dT', \alpha \in K$.

So we must prove that

$$\omega_0 \wedge \frac{df}{f} \sim 0 \Rightarrow \overline{\omega}_0 \sim 0$$

i.e., that the first condition implies that ω_0 is a coboundary modulo $I = (\mathbf{p}, d\mathbf{p})$. However, modifying ω_0 modulo dI, we may assume $\omega_0 = \alpha dT', \alpha \in K$. And, if

$$\alpha dT' \wedge \frac{df}{f} = d(\sum_i \alpha_i dT'_{[i]} \wedge \frac{df}{f})$$

with all $\alpha_i \in K$, then, obviously,

$$\alpha dT' \equiv d(\sum_i \alpha_i dT_{[i]}) \pmod{I,\, dI}$$

E.IX.6 Example:

Let $A = k[x, y] = k[T, U]$, $f(x, y) = x^2 + y^2 - 1)$.

Using U as parameter variable and T as fiber variable we get

$$\left[\frac{dTdUe^{Tx+Uy}}{f^2}\right]_{(T,U)=(0,0)} = \left[\frac{df dUe^{Tx+Uy}/2T}{f^2}\right] = \left[\frac{dTdU(1 - Tx)e^{Tx+Uy}/2T^2}{f}\right] =$$

$$= \left[\frac{dUdf(Tx - 1)e^{Tx+Uy}/4T^3}{f}\right] = \frac{x\tau - 1}{4\tau^3} e^{\tau x + \mu y} d\mu$$

where we used the exterior differentiation formula ("partial integration") in the second equality:

$$\frac{df\,dU\,e^{Tx+Uy}}{2Tf^2} = d(\frac{-1}{2Tf}dU\,e^{Tx+Uy}) + (\frac{1-Tx}{2T^2f})dT\,dU$$

Using T as parameter and U as fiber variable we get instead

$$\frac{1-y\mu}{4\mu^3}e^{\tau x+\mu y}d\tau$$

and the two are indeed cohomologous. Their difference is the coboundary of

$$-\frac{e^{\tau x+\mu y}}{4\tau\mu}$$

This idea, viewing the residue as a cohomology class, goes back to Leray, cf. e.g., [BRC].

Remark: In this form the pre-residue may be interpreted as the homomorphism $d : E(A/(0)) \to E(A/(f))$ entering an injective resolution of A. Again, this construction may be generalized to any dimension/codimension, using fundamental classes.

E.IX.7 Idempotents Revealed. A Taylor Expansion.

Refer back to the notation of III.3.-5. "Res" will denote the (scalar) pre-residue, "res" the Grothendieck residue, i.e., the pre-residue followed by $\mathrm{Tr} : K \to k(\tau')$.

$\mathbf{q} = (Q(f))$ will denote a $\mathbf{p}$–primary ideal, locally a complete intersection at $\mathbf{p}$, i.e., $\mathbf{q}S = (g_{d+1}, \ldots g_n) = (g)$.

Expressing the $f(x) - f(T)$ in the $x'' - T''$ as column matrices, $(f) = M_0(x'' - T'')$, we get the same determinant $D_0 = \det M_0$, as in III.2.-III.3. and the Wiebe Theorems hold.

We we will still write $(q(x'' - \tau''))$ for the $(x'' - \tau'')$-primary component of $\mathbf{q}$ at $(x'' - \tau'')$ in the ring $K[x'']$.

Similarly, we have expressions

$$(g(\tau', x'') - g(\tau', \tau'')) = M(\tau'', x'')(x'' - \tau'')$$

and a determinant $D(x'', \tau'')$, and, again, the Wiebe Theorems.

Recall the meaning of the idempotents ϵ and ϵ_0 from III.3.-5. It is fairly easy to prove that $\epsilon \equiv \epsilon_0 \pmod{(f(\tau', x''))}$.

Consider an element $h(\tau', x'') \in S/(g) \simeq k(\tau')[x'']/(g) \subset K[x'']/(g(\tau', x''))$. We are looking for a "Taylor expansion" of h. It is natural to expand $H = \epsilon h$ in monomials in $x'' - \tau''$ –since ϵh may be conceived of as an element of $K[x'' - \tau'']/(q(x'' - \tau''))$ – and then take $\mathrm{Tr}(\epsilon h) \in k(\tau')[x'']/(g)$.

We will presently give the expansion as a residue. By cofinality, $\mathrm{Ext}_S^{n-d}(S/(g), S)$ fits into the direct system of Ext–modules used to define the pre-residue, so we may define the pre-residue

$$\mathrm{Res}\begin{bmatrix} h(T)dT \\ g(T) \end{bmatrix}$$

The non-degeneracy property established in IX.3. still holds.

Since for any locally complete intersection $(k) \supset (g)$ the canonical projection $S/(g) \to S/(k)$ induces a determinant mapping

$$\Delta(T) : \mathrm{Ext}_S^{n-d}(S/(g), S) \to \mathrm{Ext}_S^{n-d}(S/(k), S)$$

we have the determinantal transition formula

$$\mathrm{Res}\begin{bmatrix} h(T)dT \\ g(T) \end{bmatrix} = \mathrm{Res}\begin{bmatrix} h(T)\Delta(T)dT \\ k(T) \end{bmatrix}$$

We henceforth fix the notation Δ for the determinant belonging to the inclusion $(g) \subset (f)$.

Let us put

$$\epsilon(x'', \tau'')m_j(x'') = \sum_k \alpha_{j,k} m_k(x'') \in K[x'']/(g(\tau', x''))$$

with all $\alpha_{jk} \in K = k(\tau')[\tau'']$. Here the $m_j(\tau', x'') \in A/\mathbf{q}$ are a K–basis of $k(\tau', x'')[x'']/(g(\tau', x'')$, or, equivalently, a K–basis of $K[x'']/(q(x' - \tau''))$. In the first case, K refers to the Cohen subfield. Recall the meaning of the symbol $\widetilde{x}''$, with $f_i(\tau', \widetilde{x}'') = 0, i = d+1, \ldots, n$

We wish to find $n_j^*(T)$ such that

$$\mathrm{Res}\begin{bmatrix} \sum_j m_j(x'')n_j^*(T'')h(T'')dT \\ g(T) \end{bmatrix} \equiv \epsilon(x'', \tau'')h(x'') \qquad (*)$$

Their existence follows from the non-degeneracy of the pre-residue form Φ. Writing $h(x'')$ as a K-linear combination of the $m_j(x'')$, with coefficients $h_i(\tau', \widetilde{x}'')$, we see that we get by with

$$n_k^*(T'') = \sum_k \alpha_{jk} m_j^*$$

where the $m_j^*(T)$ are a dual basis (for Φ) to the $m_j(T)$.

It would seem that, by K–linearity, we will get $h_i(\tau', \tau'')$ in place of $h_i(\tau', \widetilde{x}'')$ in the expression outside the residue symbol (since we get $h(T', \widetilde{T}'') \pmod{(g(T))}$ inside it.) This, however, makes no difference since, by II.7.,

$$\epsilon = \frac{D_0(\tau'', \widetilde{x}'')}{D_0(\tau'', \tau'')}$$

and

$$D_0(\tau'', \widetilde{x}'')(\widetilde{x}'' - \tau'') \subset (f(\widetilde{x}'') - f(\tau'')) = (0 - 0) = (0)$$

Setting $h = 1$ we see that

$$\epsilon = \operatorname{Res}\left[\begin{array}{c} \sum_j m_j(x'')n_j(T'')dT \\ g \end{array}\right]$$

Let $S(x'', T'')$ denote the sum in the numerator.

We then see that $T_i''h(T), x_i''h(T)$ and $x_i''h(x'')$ yield the same result $\epsilon x_i''h(x'')$ for any h, so

$$S \in (g) : (x'' - T'')$$

i.e., S is a multiple $\alpha(T'', x'')D(x'', T'')$ of $D(x'', T'')$. Here we may replace $\alpha(T'', x'')$ by $\beta(x'') = \alpha(x'', x'')$.

Our aim is to show that $S(x'', \tau'') \equiv D(x'', \tau'') \pmod{g(x'')}$, i.e., $\beta(x'') \equiv 1 \pmod{g(x'')}$. For this we need only prove that

$$\delta(x'', \tau'') = \operatorname{Res}\left[\begin{array}{c} D(x'', T'')dT \\ g(T) \end{array}\right]_{T=\tau} \equiv 1 \pmod{q(x'' - \tau'')}$$

where $q(x'' - \tau'')$ denotes the $(x'' - \tau'')$–primary component of $(g) = Q(f)$. On account of the following lemma this is the same as proving that

$$h(x'', \tau'') = \delta \text{ and } h(x'', \tau'') = \delta' = \operatorname{Res}\left[\begin{array}{c} D(\tau'', T'')dT'' \\ g(T) \end{array}\right]$$

map the same under all Noetherian operators

$$\operatorname{Res}\left[\begin{array}{c} h(x'', \tau'')m(x'')dx'' \\ (q(x'' - \tau'')) \end{array}\right]_{x'' - \tau''}$$

E.IX.8 Lemma.

$$\delta(\tau'', \tau'') = \operatorname{Res}\left[\begin{array}{c} D(\tau'', T'')dT \\ g(T) \end{array}\right]_{T=\tau} = 1$$

Proof:

$$\text{Res}\left[\begin{array}{c} D(\tau'',T'')dT \\ g(T) \end{array}\right] = \text{Res}\left[\begin{array}{c} \Delta(T)D_0(\tau'',T'')dT \\ g(T) \end{array}\right] =$$

$$= \text{Res}\left[\begin{array}{c} D_0(\tau'',T'')dT \\ f(T) \end{array}\right]$$

by the determinantal transition formula. And this in turn equals

$$\text{Res}\left[\begin{array}{c} D_0(\tau'',T'')dT'/J\ df \\ f(T) \end{array}\right] = (\frac{D_0(\tau'',T'')}{J})_{T''=\tau''} = \frac{D_0(\tau'',T'')}{D_0(\tau'',\tau'')}_{T''=\tau''} = 1$$

▮

We now turn to the determination of

$$\text{Res}\left[\begin{array}{c} \delta(x'',\tau'')m(x'')dx'' \\ (q(x''-\tau'')) \end{array}\right]_{x''=\tau''}$$

The actions of the two Noetherian pre-residues, w.r.t to T'' and x'' respectively, commute and may be put together into a super-residue ("transitivity formula")

$$\text{Res}\left[\begin{array}{c} D(x'',T'')m(x'')dT\wedge dx'' \\ g(T)\,(q(x''-\tau'')) \end{array}\right]_{T=x''=\tau}$$

By the determinant formula, $g(T)$ may be modified by anything in $(q(x''-\tau''))$, e.g., $g(x'')$. So, by another invocation of the determinant formula we obtain

$$\text{Res}\left[\begin{array}{c} D(x'',T'')m(x'')d(T''-x'')\wedge dx'' \\ (g(\tau',T'')-g(\tau',x''))\,q((x''-\tau'')) \end{array}\right] = \text{Res}\left[\begin{array}{c} dT'd(T''-x'')m(x'')dx'' \\ (T''-x'')\,(q(x''-\tau'')) \end{array}\right] =$$

$$\text{Res}\left[\begin{array}{c} m(x'')dx'' \\ (q(x''-\tau'')) \end{array}\right]_{x''=\tau''}$$

With δ' in place of δ we get

$$\text{Res}\left[\begin{array}{c} D(\tau'',T'')m(x'')dT\wedge dx'' \\ g(T)\,(q(x''-\tau'')) \end{array}\right] = \text{Res}\left[\begin{array}{c} D_0(\tau'',T'')\Delta(T)m(x'')dTdx'' \\ g(T)\,(q(x''-\tau'')) \end{array}\right] =$$

$$\text{Res}\left[\begin{array}{c} D_0(\tau'',x'')m(x'')dT\wedge dx'' \\ f(T)\,(q(x''-\tau'')) \end{array}\right] = \text{Res}\left[\begin{array}{c} m(x'')dx'' \\ (q(x''-\tau'')) \end{array}\right]_{x''=\tau''}$$

where the last step is just as in the proof of the Lemma.

So, finally we have proved:

E.IX.9 Theorem. *Assumptions as above.*

a) The idempotent ϵ *of III.3.-5. is given by*

$$\epsilon = \text{Res}\left[\begin{array}{c} D(T'', x'')dT \\ g \end{array}\right]_{T''=\tau''}$$

b) The "Taylor expansion" of $h(\tau', x'')$ *modulo* $(g(\tau', x''))$ *is given by*

$$\text{res}\left[\begin{array}{c} D(T'', x'')h(T)dT \\ g(T) \end{array}\right] =$$

$$\text{Tr}_{K/k(\tau')}\,\text{N}_0(\tau, \frac{\partial}{\partial T''})(D(x'', T'')h(x', T''))_{T''=\tau''} =$$

$$\text{Tr}_{K/k(\tau')}\,D(\tau'', \frac{\partial}{\partial x''})[h(\tau', x'')\text{N}_0(\tau, x'')e^{\tau x}]_{x=0}$$

(recall that "res" is the trace of "Res")

Here N_0 denotes the generating Noetherian operator given by the pre-residue.

Remark: By the equality

$$D(T'', x'')(T'' - x'') = M^*(T'', x'')(g(T'') - g(x''))$$

one easily proves: For any Noetherian operator $N(\partial/\partial x'')_{x''=\tau''}$ killing $g(x'')$ at (τ) it holds that $N(D) = 0$ at any conjugate $(\tau)^g; g \neq 1$ of (τ) (over $k(\tau')$. That is, $D(x'', \tau'')$ belongs to all the conjugates of $q(x'' - \tau'')$).

The proof uses Leibniz' rule (since both members contain products) along with the elementary determinantal criterion for singularity of a matrix, and proceeds by induction on the degree of the Noetherian operator. This idea elaborates on one of Kronecker. A quicker proof proceeds along the lines of the computation preceding the Theorem, where the last "super-residue" (at $T'' = \tau''^g, x'' = \tau''$) is easily seen to equal 0.

This observation yields an alternative to the rather subtle introductory arguments, and the reliance on Theorem II.7.

Using the Reciprocity Theorem, and all that precedes it, one may also prove:

E.IX.10 Theorem. *Consider the system*

$$g_i(\frac{\partial}{\partial x})p(\tau', x'')e^{\tau' x'} = 0 \quad i = d+1, \dots, n$$

("true" exponentials) with the "initial data"

$$\mathcal{D}(X'', \frac{\partial}{\partial x''})(p(\tau', x'')e^{\tau' x'})_{x''=0} = \sum_i (m_i(X'')m_i^*(\frac{\partial}{\partial x''})(p(\tau', x'')e^{\tau' x'})_{x''=0} =$$

$$\sum_i m_i(X'')n_i(\tau')e^{\tau' x'};$$

where the $m_i(x'')$ are a $k(\tau')$–basis of $k(\tau', x'')/(g(\tau', x''))$

Then the solution is given by

$$p(\tau', x'')e^{\tau x} = \mathrm{res}\begin{bmatrix} \sum_i m_i(T'')n_i(T')e^{Tx}dT \\ g(T) \end{bmatrix}$$

Taking the pre-residue instead we get the generic exponential mapping to $p(\tau', x'')e^{\tau x}$ under the trace mapping.

(Here the X'' are indeterminates allowing as to write several equations as one.)

We also note the following corollary

E.IX.11 Corollary. *The true exponential*

$$e_0(\tau', x'') = \mathrm{res}\begin{bmatrix} e^{Tx}dT \\ g(T) \end{bmatrix}$$

is that solution $p = e_0$ of the system $g_i(\partial/\partial x)p = 0$ for which the "initial data"

$$D(X'', \frac{\partial}{\partial x''})(e_0)_{x=0} = 1$$

("impulse response"; all derivatives, except "the highest", $= 0$).

∎

Theorem IX.10. is an exact analogue of the one-variable case, which is usually found by means of Laplace transforms. I do not think it is of any great interest except in the zero-dimensional case, where pre-residues and residues are the same thing (as is the case with all linear schemes).

The Theorem can be generalized to situations involving more varieties than one, e.g., the ones linked to **p** by the prime sequence (g), which again requires a more global residue and fundamental classes.

E.IX.12 Example:

Refer back to Examples IX.6 and III.6. The solutions

$$\frac{x\tau - 1}{4\tau^3} e^{\tau x + \mu y} d\mu \quad \text{and} \quad \frac{1 - y\mu}{4\mu^3} e^{\tau x + \mu y} d\tau$$

found in that Example correspond to the impulse response solutions belonging to the respective normalizations.

The idempotent in III.6. is produced by the pre-residue operator

$$\frac{-1 + \tau \frac{\partial}{\partial T}}{4\tau^3}$$

acting on

$$\frac{f(x)^2 - f(T)^2}{x - T} = xf(x) + Tf(x) + xf(T) + Tf(T)$$

followed by the evaluation $T = \tau$, (modulo $f^2(\tau', x'')$, of course), as promised by our Theorem.

∎

The restriction of Theorem E.IX.10 to true exponentials is maybe somewhat artificial. Here is a more general statement

E.IX.13 Theorem. *Consider the system*

$$g_i(\frac{\partial}{\partial x}) y(x', x'') = 0; \qquad i = d+1, \ldots, n$$

with "initial data"

$$\mathcal{D}(X'', \frac{\partial}{\partial x''})(y(x', x''))_{x''=0} =$$

$$\sum_i m_i(X'') y_i(x')$$

(where the y_i are, e.g., formal series). Let

$$f_i(\tau', x'') = \operatorname{res} \begin{bmatrix} m_i(T'') e^{Tx} dT \\ g(T) \end{bmatrix}$$

Then a formal solution is given by

$$y(x', x'') = \sum_i f_i(\frac{\partial}{\partial x'}, x'') y_i(x')$$

We illustrate the Theorem with a simple example

E.IX.14 Example:

The follwing Example is due to Gröbner.

We will work in

$$k[x_1, x_2]/(x_1^2 + x_2^2) = k[\tau_1', \tau_2'']$$

where

$$\mathrm{Tr}_{K/k(\tau_1')}\, \tau_2^{2k+1} = 0$$

and

$$\mathrm{Tr}_{K/k(\tau_1')}\, \tau_2^{2k} = 2 \cdot (-1)^k \tau_1^{2k}$$

since $\tau_2^2 = -\tau_1^2$

Consider the equation

$$\begin{aligned} (x_1^2 + x_2^2) * y(x_1', x_2'') &= 0 \\ y(x_1, 0) &= y_1(x_1) \\ \frac{\partial y}{\partial x_2}(x_1, 0) &= y_2(x_1) \end{aligned}$$

Here

$$\mathcal{D}(X'', \frac{\partial}{\partial x''}) = X_2 + \frac{\partial}{\partial x_2}$$

and

$$\mathrm{res} \left[\begin{array}{c} (X_2 + T_2)e^{Tx} dT \\ T_1^2 + T_2^2 \end{array} \right] = \mathrm{Tr}_{K/k(\tau_1')} \frac{X_2 + \tau_2}{2\tau_2} e^{\tau_2 x_2}$$

So

$$f_1(\tau_1, x_2) = \mathrm{Tr}\, \frac{1}{2} e^{\tau_2 x_2} = \frac{1}{2}(e^{\tau_2 x_2} + e^{-\tau_2 x_2}) =$$

$$= \sum_k (-1)^k \frac{\tau_1^{2k} x_2^{2k}}{(2k)!}$$

$$f_2(\tau_1, x_2) = \mathrm{Tr}(\frac{1}{2\tau_2} e^{\tau_2 x_2}) = \frac{1}{2\tau_2}(e^{\tau_2 x_2} - e^{-\tau_2 x_2}) =$$

$$= \sum_k (-1)^k \frac{\tau_1^{2k} x_2^{2k+1}}{(2k+1)!}$$

and

$$y(x_1, x_2) = f_1(\partial/\partial x_1, x_2) y_1 + f_2(\partial/\partial x_1, x_2) y_2$$

s

E.X Examples of Noetherian Operators

E.X.1 Example: A 0-dimensional Scheme

This example is taken from [EHR].

Consider the (x_1, x_2)- primary ideal

$$\mathbf{q} = (f(x), g(x)) = (x_1^2 - x_2, x_2^2) \supset (F, G) = (x_1^4, x_2^2);$$

in $A = k[x_1, x_2]$. The easiest way to compute the Notherian operators belonging to this ideal is to compute the Taylor expansion:

$$f(\mu_1, \mu_2) = e^{\mu_1 D_1 + \mu_2 D_2} f(0,0)$$

Here μ_i denotes the class of x_i modulo $\mathbf{q}$; D_i is $\partial/\partial x_i$.
Using

$$\mu_1^4 = \mu_2^3 = \mu_1^3 - \mu_1\mu_2 = 0$$

and the easily verified k-linear independence of $1, \mu_1, \mu_2, \mu_1\mu_2$, we find

$$e^{\mu_1 D_1 + \mu_2 D_2} = (1 + \mu_1 D_1 + \frac{1}{2}\mu_1^2 D_1^2 + \frac{1}{6}\mu_1^3 D_1^3)(1 + \mu_2 D_2) =$$

$$\mu_1\mu_2(\frac{1}{6}D_1^3 + D_1 D_2) + \mu_2(\frac{1}{2}D_1^2 + D_2) + \mu_1 D_1 + 1 \qquad (*)$$

From this we see that the k-vector space of Noetherian operators is generated by

$$D = [\frac{1}{6}\frac{\partial^3}{\partial x_1^3} + \frac{\partial^2}{\partial x_1 \partial x_2}]_{(0,0)}$$

along with

$$D^{(1)} = [D, x_2] = \frac{1}{2}D_1^2 + D_2; \quad D^{(2)} = [D, x_1]; \quad D^{(12)} = [D^{(1)}, x_2] = [D^{(2)}, x_1]$$

all evaluated at $(0,0)$

E.X.2 Example: The same example reciprocated

Using our previous identification way may look at the solutions of the system

$$f(\frac{\partial}{\partial x})p = g(\frac{\partial}{\partial x})p = 0$$

One solution is

$$e_0 = \frac{1}{6}x_1^3 + x_1 x_2$$

and the remaining solutions may be found on repeated differentiation. Substituting x_i for D_i in the expression $(*)$ we can read off a basis for the solution space as the coefficients in

$$(\frac{1}{6}x_1^3 + x_1x_2)\mu_1\mu_2 + (\frac{1}{2}x_1^2 + x_2)\mu_2 + x_1\mu_1 + 1$$

which may also be written

$$\mathcal{D}(\mu, \frac{\partial}{\partial x})e_0 = (\frac{\partial^2}{\partial x_1 \partial x_2} + \mu_1\frac{\partial}{\partial x_2} + \mu_2\frac{\partial}{\partial x_1} + \mu_1\mu_2 \cdot 1)e_0$$

Here, $\mathcal{D}(\mu, x)$ is the determinant of the matrix $\mathcal{M}(\mu, x)$ in

$$\begin{pmatrix} f(x) - f(\mu) \\ g(x) - g(\mu) \end{pmatrix} = \mathcal{M}(\mu, x)\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix};$$

$$\begin{pmatrix} x_1^2 - x_2 - \mu_1^2 + \mu_2 \\ x_2^2 - \mu_2^2 \end{pmatrix} = \begin{pmatrix} x_1 + \mu_1 & -1 \\ 0 & x_2 + \mu_2 \end{pmatrix}\begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

and e_0 is the unique solution of the system above satisfying

$$\mathcal{D}(\mu, \frac{\partial}{\partial x})e_0(0, 0) = 1$$

i.e., the "impulse response" of IX. 11.

Another invocation of Wiebe's Theorem shows that

$$\begin{pmatrix} x_1^4 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1^2 - x_2 \\ x_2^2 \end{pmatrix}$$

with determinant $d(x) = x_1^2 + x_2$, hence $(x_1^4, x_2^2) : (x_1^2 - x_2, x_2^2) = x_1^2 + x_2$. The impulse response solution for the system $F(\partial/\partial x)p = G(\partial/\partial x)p = 0$ is easily found to be

$$E_0 = \frac{1}{6}x_1^3x_2$$

The duality theory may be used to explain why

$$e_0 = d(\frac{\partial}{\partial x})E_0$$

By Theorem E.IX.4. we see that the solution space of the system $(F, G) * p = 0$ is mapped onto the the solution space of $(f, g) * p = 0$ by the differential operator $d(\partial/\partial x)$

Also, let $\mathcal{D}_1(x, \mu)$ denote the determinant of the square matrix $\mathcal{M}_1$ in

$$\begin{pmatrix} x_1^4 \\ x_2^2 \end{pmatrix} = \mathcal{M}_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

From the obvious fact that

$$\mathcal{D}_1(0, x) = d(x)\mathcal{D}(0, x)$$

and the fact that all terms of $\mathcal{D}_1(\mu, x)$ except the constant are derivatives of the impulse response one easily sees that

$$\mathcal{D}(\mu, \frac{\partial}{\partial x})d(\frac{\partial}{\partial x})E_0 = 1$$

as claimed.

E.X.3 Example: A Linear 1-dimensional Scheme

This example is discussed in [HÖR], [BJÖ], and [PAL].

Here we discuss the primary ideal $\mathbf{q} = (x_3^2,\ x_2 - x_1x_3)$ belonging to the prime ideal $\mathbf{p} = (x_2, x_3) \in k[x_1, x_2, x_3]$.

Again, the easy way out is to compute the Taylor expansion

$$e^{\mu D} = e^{\mu_1 D_1}(1 + \mu_2 D_2)(1 + \mu_3 D_3) =$$

$$e^{\mu_1 D_1}(1 + \mu_1\mu_3 D_2)(1 + \mu_3 D_3) =$$

$$e^{\mu_1 D_1}(1 + \mu_1\mu_3 D_2 + \mu_3 D_3)$$

from which we read off the operators

$$1; \quad \tau_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$$

(at $(\tau_1, 0, 0)$) generating the Noetherian operators as vector space over $k(\tau_1')$.

E.X.4 Example: A Non-linear Example

Let us look at the prime ideal $\mathbf{p}$ defining the affine cone over the twisted cubic, $\mathbf{p} = (s^3, s^2t, st^2, t^3) \in A = k[x_0, x_1x_2, x_3]$, It is well-known that $\mathbf{p}$ possesses the following generators:

$$e = x_1^2 - x_0x_2; \quad f = x_1x_2 - x_0x_3; \quad g = x_2^2 - x_1x_3$$

Also $\mathbf{p}A_{\mathbf{p}} = (e, g)A_{\mathbf{p}};\ (e, g) : \mathbf{p} = (x_1, x_2)$

Let us look at the ideal

$$\mathbf{q} : (e = x_1^2 - x_0x_2, \quad h = x_3f - x_2g = -x_0x_3^2 + 2x_1x_2x_3 - x_2^3)$$

which is easily seen to contain $\mathbf{p}^2$. To see that it is $\mathbf{p}$-primary set the two generators equal to zero:

$$x_3f - x_2g = 0$$

$$e = 0; x_3e = 0 \quad x_3(x_1^2 - x_0x_2) = x_2f - x_1g = 0$$

where we used $x_3e - x_2f + x_1g = 0$

If this does not entail $f, g = 0$, then the determinant $x_2^2 - x_1x_3 = 0$, i.e., $g = 0$.

Using the relation $x_2e - x_1f + x_0g = 0$ we similarly find (on multiplying $e = 0$ by x_2) that $f = 0$, contradiction.

Since $\mathbf{q} \supset \mathbf{p}^2$ we should look for first-order operators characterizing $\mathbf{q}$. They are easily found from the fact that the linear parts of e and h, at $(\tau_0', \tau_1'', \tau_2'', \tau_3')$, are linearly dependent. Writing $u_1 = x_1'' - \tau_1''; u_2 = x_2'' - \tau_2''$ we find

$$e(\tau', x'') - e(\tau', \tau'') = 2\tau_1 u_1 - \tau_0 u_2 + \text{ higher terms}$$

$$h(\tau', x'') - h(\tau', \tau'') = \tau_2(-2\tau_3 u_1 + \tau_2 u_2) + \text{ higher terms}$$

which are linearly dependent because the determinant $2(\tau_1\tau_2 - \tau_0\tau_3) = 0$

The obvious choice for a Notherian operator killing $\mathbf{q}$ is therefore

$$\tau_2 \frac{\partial}{\partial x_1} + 2\tau_3 \frac{\partial}{\partial x_2}$$

at $(\tau_0', \tau_1'', \tau_2'', \tau_3')$. It will kill the linear parts and the higher parts as well (on evaluation), since their first derivatives are of positive degree in the u_1, u_2.

The possible commutators are constants.

E.X.5 Example: A Reducible 0-dimensional Example

This example elaborates on an example in [BEA]

Consider the ideal

$$(F, G) = (x_1^3 + x_2^2,\ x_1x_2 - 2x_2^3) \subset (x_1, x_2) \subset k[x_1, x_2]$$

It is not (x_1, x_2)- primary, but actually has five different primary components. We wish to find the (x_1, x_2)-primary component.

First of all, we note

$$\begin{pmatrix} x_1^3 + x_2^2 \\ x_1x_2 - 2x_2^3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_2 \\ x_2 & -2x_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The determinant of the square matrix is $D = -x_2^2(1 + 2x_1^2)$

By Wiebe's Theorem

$$(F, G) : D = (x_1, x_2); \quad (F, G) : (x_1, x_2) = (D, F, G)$$

From the fact that x_2^2 belongs to $\mathbf{m} = (x_1, x_2)$, and $1 + 2x_1^2$ does not, one easily infers that the $\mathbf{m}$-primary part of (F, G) is

$$(F, G) : (1 + 2x_1^2) = \mathbf{q}$$

and that

$$(1 + 2x_1^2)(x_1^n, x_2^n) \subset (F, G)$$

for large enough n. Indeed,

$$(1 + 2x_1^2) \begin{pmatrix} x_1^4 \\ x_2^3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_1^3 - 2x_2^2 & -x_2 \\ x_2 & -x_1^2 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \qquad (*)$$

The determinant of the square matrix is $(-x_1^3 + x_2^2)(1 + 2x_1^2)$

Now, in this case a generating Noetherian operator is given by the residue symbol as explained in E.IX.3:

$$\mathrm{Res}_{(0,0)}\left[\begin{array}{c} p(1+2x_1^2)dx_1dx_2 \\ F\ G \end{array}\right] =$$

$$\mathrm{Res}_{(0,0)}\left[\begin{array}{c} (-x_1^3+x_2^2)(1+2x_1^2)^2/(1+2x_1^2)^2 p dx_1 dx_2 \\ x_1^4\ x_2^3 \end{array}\right] = (-\frac{1}{2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{6}\frac{\partial^3}{\partial x_1^3})p(0,0)$$

Here we used the "determinantal formula" given in any account of residues ([SS], [KRD], [DIS], [BEA]), and, implicitly, in IX.7, and Example X.2. above. The denominator $(1+2x_1^2)^2$ comes from the left member of $(*)$

This operator, along with its commutators, characterize the ideal $\mathbf{q}$. By the Reciprocity Theorem, $q \in \mathbf{q}$ if and only if

$$q(\partial/\partial x)(-3x_2^2 + x_1^3) = 0$$

whence, after some computation,

$$\mathbf{q} = (x_1^3 + x_2^2,\ x_1x_2)$$

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